DISCRETE MATH (MATH 2001)

REVIEW SHEET 2

II. Logic (continued from Review sheet 1)
  (f) Proof writing strategies.
    (i) Direct versus indirect proof.
    (ii) “Iff”, and “TFAE” proofs.
    (iii) Case division.
    (iv) Quantifiers.
    (v) When are examples and counterexamples appropriate?

III. Induction
  (a) Ordinary induction.
  (b) Strong induction.
  (c) Recursive definitions of arithmetic operations on $\mathbb{N}$: $x + y, xy, x^y$.
  (d) Use of induction to prove laws of arithmetic.

IV. Relations
  (a) Ordered pairs, triples and n-tuples. $A \times B$.
  (b) Definitions of *relation* and *function*. Representations of relations and functions.
  (c) Injections, surjections, bijections.

V. The Canonical Factorization of a Function
  (a) Domain and codomain.
  (b) Kernel. Equivalence relation.
  (c) Coimage. Partition.
  (d) Natural map. Inclusion map. Induced map. $f = \mathrm{inc} \circ \overline{f} \circ \mathrm{nat}$.
  (e) Well-defined functions.

VI. Counting
  (a) Definitions of $|A| = |B|$, $|A| \leq |B|$, $|A| < |B|$, $|A| = m$, finite and infinite, countable and uncountable.
  (b) Sum Rule and Product Rule.
  (c) Simple counting formulas:
    (i) There are m^n functions from an n-element set to an m-element set.
    (ii) There are $(m)_n = m \cdot (m-1) \cdots (m-n+1)$ injective functions from an n-element set to an m-element set. There are $n!$ bijections between two n-element sets.
    (iii) There are $n!$ ways to linearly order an n-element set.
  (d) Binomial coefficients.
    (i) There are $\binom{n}{k}$ k-elements subsets of an n-element set.
    (ii) Binomial Theorem.

(iii) Pascal's identity. Pascal's triangle: unimodality and symmetry of nth row and nth row sum.
(iv) ~~Generalization to multinomial coefficients: there are $\binom{n}{k_1,\ldots,k_r}$ ordered partitions of n into cells of sizes $(k_1, \ldots, k_r)$; Multinomial Theorem; Pascal's Pyramid; nth row sum.~~

(e) Counting multisets.
(f) ~~Principle of Inclusion and Exclusion: Formula.~~ Counting surjective functions. Stirling numbers of the second kind.
(g) Distribution problems.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(1) Know the definitions of new concepts, and the meanings of the definitions.
(2) Know the statements and meanings of the axioms and major theorems.
(3) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(4) Know how to perform the different kinds of calculations discussed in class.
(5) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(6) Know how to correct mistakes made on old HW.

More specific advice.
Be prepared to demonstrate understanding in the following ways.

(1) Know the definitions of the following: contrapositive; converse; ordered pair; $A \times B$; relation; the following properties of binary relations: reflexive, irreflexive, symmetric, antisymmetric, asymmetric, transitive; equivalence relation; function; kernel; image; preimage; coimage; natural map; inclusion map; induced map; injection; surjection; bijection; $|A| = |B|$; $|A| \leq |B|$; $|A| < |B|$; $|A| = m$; finite and infinite; countable and uncountable; binomial coefficients; multiset; (~~ordered or unordered~~) partition.
(2) Know the statements and meanings of: the theorem proving that induction works; the Recursion Theorem; the Cantor-Schroeder-Bernstein Theorem; Cantor's Theorem; Sum Rule; Product Rule. Binomial Theorem; ~~Principle of Inclusion and Exclusion.~~
(4) Know how to: organize a proof so that it is a direct proof, a proof of the contrapositive, or a proof by contradiction; prove statements by induction; describe the canonical factorization of a function; determine if a function is well-defined; establish that $|A| = |B|$, $|A| \leq |B|$, $|A| < |B|$, $|A| \neq |B|$, etc; count the number of all functions, injective functions or bijective functions from a set of size m to a set of size n; count the number of m-element subsets, m-element multisets or m-cell partitions of an n-elements set; apply all the counting formulas on the 'Distributions' handout.

Test your understanding.

(1) Explain why induction is a valid form of proof.

(2) Prove that $m^{n+p} = m^n m^p$ for all $m, n, p \in \mathbb{N}$.

(3) Suppose you want to prove a theorem with two hypotheses: $((H_1 \wedge H_2) \to C)$. Which of the following proof strategies would *suffice* to prove the theorem? Explain your answer.
   (i) A proof of $(((\neg C) \wedge H_2) \to (\neg H_1))$ would suffice.
   (ii) A proof of $((\neg H_1) \vee (\neg H_2) \vee C)$ would suffice.
   (iii) A proof of $((\neg C) \to ((\neg H_1) \wedge (\neg H_2)))$ would suffice.

(4) What is a function? (Give the definition.) If f is a function, under what circumstances will f^{-1} also be a function?

(5) If $U \subseteq A$, then the *characteristic function* of U is the function $\chi_U \colon A \to \{0, 1\}$ defined by $\chi_U(a) = 1$ if $a \in U$ and $\chi_U(a) = 0$ if $a \notin U$.
   (i) Explain why every function from A to $\{0, 1\}$ is the characteristic function of some subset $U \subseteq A$.
   (ii) Show that if K is the set of all characteristic functions defined on A, then $|K| = |\mathcal{P}(A)|$.
   (iii) Explain why $|\mathcal{P}(A)| = 2^{|A|}$.

(6) How many ways are there to make a circular necklace with n beads of different colors if two necklaces are considered to be the same if they differ by a rotation? What if two necklaces are considered to be the same if they differ by a rotation or a flip?

(7) ~~What is the constant term in~~ $\cancel{(x^{-2} + 2x^{-1} + 3 + 5x)^3}$~~?~~

(8) You have just given birth to octuplets. How many ways can you name your children if you only like the names Billy Bob, Jim Bob and Sue Bob?

(9) If you deal a random 2-card hand, what is the probability of blackjack? (An ace together with a 10 or face card.)