Solutions to HW 2.

Initial remarks:

- "Prove", "show" and "explain" all mean the same thing: communicate reasons in a clear and precise way.
- A formal proof of a statement S is a list of statements, say $S_0, S_1, S_2, \ldots, S_n = S$, where each S_i follows from earlier statements by accepted rules of deduction and the last statement is S. Most written proofs are not formal; rather, they are written informally in paragraph form. Nevertheless, an informal proof is correct only if it is clear how to transcribe it into a formal proof.
- In a "yes/no" question, the first sentence of the solution should indicate whether the answer is yes or no. Similarly, in a "prove or disprove" question, you should indicate in the first sentence whether or not you are proving are disproving the statement.
- Usually, to prove a statement you must supply an argument, while to disprove a statement it is enough to supply a counterexample.

1. Prove or disprove:
(i) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
(ii) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

Item (i): This statement is true. We show that $\mathcal{P}(A \cap B)$ and $\mathcal{P}(A) \cap \mathcal{P}(B)$ are equal by showing they have the same elements. For this we need to check two things:
(a) If $X \in \mathcal{P}(A \cap B)$, then $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$.
(b) If $Y \in \mathcal{P}(A) \cap \mathcal{P}(B)$, then $Y \in \mathcal{P}(A \cap B)$.

Proof of (a): Choose $X \in \mathcal{P}(A \cap B)$. Then $X \subseteq A \cap B$ (definition of power set). Hence any $t \in X$ satisfies $t \in A \cap B$ (definition of subset). Hence any $t \in X$ satisfies $t \in A$ and $t \in B$ (definition of intersection). This implies that if $t \in X$, then $t \in A$, and it also implies that if $t \in X$, then $t \in B$; hence $X \subseteq A$ and $X \subseteq B$ (definition of subset). This proves that $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$ (definition of power set). Hence $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$ (definition of intersection).

Proof of (b): Just reverse the order of statements of the proof of (a) using Y in place of X.

Item (ii): This statement is false. Let $A = \{2\}$ and let $B = \{7\}$. Then

$$\mathcal{P}(A \cup B) = \mathcal{P}(\{2, 7\}) = \{\emptyset, \{2\}, \{7\}, \{2, 7\}\}$$

while

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{2\}\} \cup \{\emptyset, \{7\}\} = \{\emptyset, \{2\}, \{7\}\}.$$

These are different since $\{2, 7\} \in \mathcal{P}(A \cup B)$, but $\{2, 7\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

2. Prove or disprove: for any A, B and X, if $A \cup X = B \cup X$ and $A \cap X = B \cap X$, then $A = B$.

Solution: The statement is true. We will prove that $A = B$ using the assumptions that $A \cup X = B \cup X$ and $A \cap X = B \cap X$. To prove that $A = B$ we need to check two things:

(a) If $s \in A$, then $s \in B$.
(b) If $s \in B$, then $s \in A$.

Proof of (a): Choose $s \in A$. Then $s \in A \cup X$ (definition of union). Hence $s \in B \cup X$ (assumption $A \cup X = B \cup X$). Hence $s \in B$ or $s \in X$ (definition of union).

Case 1: $s \in B$.
Nothing to prove, since our goal is to show $s \in B$.

Case 2: $s \in X$.
Now $s \in A$ and $s \in X$, so $s \in A \cap X$ (definition of intersection). But $A \cap X = B \cap X$ (assumption), so $s \in B \cap X$. Therefore $s \in B$ (definition of intersection).

Both cases lead to $s \in B$, so the assumption $s \in A$ does imply $s \in B$.

Now we must prove (b). Here it is enough to say "By symmetry, if $s \in B$, then $s \in A$." The reason we can say this is that the assumptions of the problem are symmetric in A and B, and when you interchange their roles you convert the proof of item (a) to a proof of item (b).

3. Show that $A \subseteq B$ if and only if $A \cap B = A$.

Because of the "if and only if", there are two directions to prove:
(a) If $A \subseteq B$, then $A \cap B = A$.
(b) If $A \cap B = A$, then $A \subseteq B$.

Proof of (a): Must show that $A \cap B = A$ using the assumption $A \subseteq B$. Choose $x \in A \cap B$. Then $x \in A$ (definition of intersection). Conversely choose $x \in A$. Since $A \subseteq B$ we have $x \in B$, too (definition of subset). Hence $x \in A \cap B$ (definition of intersection). This proves that $x \in A$ implies $x \in A \cap B$, so together with the reverse conclusion, proved above, we get $A = A \cap B$.

Proof of (b): Must show that $A \subseteq B$, using the assumption $A \cap B = A$. Choose $x \in A = A \cap B$. Then $x \in B$ (definition of intersection). Thus $x \in A$ implies $x \in B$, so $A \subseteq B$.