Commutative Algebra (MATH 6150)

Sheaves and Schemes.

These notes provide definitions not found in the book. When it seems appropriate, I will mention the motivation for the definition. This motivation is not part of the definition.

(1) (Motivation: Sheaves of germs of analytic functions)

For definitions under item (1), let X be an open subset of the complex plane. Recall that a function $f\colon X \to \mathbb{C}$ is *analytic* at a point $p \in X$ if it equals its Taylor series in an open neighborhood of p.

(a) A *function element* is a pair (f, U) where $U \subseteq X$ is open and f is analytic at each point of U.

(b) The *germ induced by a function element* (f, U) at $p \in U$ is the pair (f, p). "**A** *germ*" is a pair (f, p) for which there exists some open U containing p such that (f, U) is a function element. (We think of the germ induced by f at p as encoding the information of the Taylor series of f at p.)

(c) Let $\mathcal{S}$ be the set of all germs of analytic functions on X. There is a (surjective) function $\pi\colon \mathcal{S} \to X\colon (f, p) \mapsto p$, called *projection*.

(d) To each function element (f, U) there is a function $\hat{f}\colon U \to \mathcal{S}\colon p \mapsto (f, p)$, which encodes (f, U), and which is a right inverse to π on U. In general, right inverses to surjective maps are called *sections* of those maps, so we call $\hat{f}$ the *section* of π associated to f. Conversely, to each function $g\colon U \to \mathcal{S}$ that is a section of π there is a function $\check{g}\colon U \to \mathbb{C}$ defined by: $\check{g}(p) = \alpha$ iff $g(p) = (f, p)$ and $f(p) = \alpha$. Notice that if you first 'hat' f, then 'check' the result, you obtain f back. This does not work in the reverse order, because any function is the 'check' of some section of $\mathcal{S}$ over U. To make this correspondence a bijection we topologize $\mathcal{S}$ and deal only with continuous sections over U.

(e) The topology on $\mathcal{S}$ is the strongest topology on $\mathcal{S}$ that makes each section $\hat{f}\colon U \to \mathcal{S}$, which arises from a function element (f, U), a continuous map. A basis for this topology turns out to be the set $\{\hat{f}(U) \mid (f, U) \text{ a function element}\}$. We proved in class that a section $g\colon U \to \mathcal{S}$ is continuous with respect to this topology iff there is an open cover $\{U_i \mid i \in I\}$ of U such that $g|_{U_i} = \hat{f}_i$ for function elements (f_i, U_i), $i \in I$. That is, a section over U is continuous iff it is the section induced by a function that is locally a function element on U. Since any function that is locally analytic on U is in fact analytic on U, this shows that the continuous sections are exactly those induced by function elements.

(f) The *stalk* at $p \in X$ is $\pi^{-1}(p) =$ the set of germs of analytic functions at p. Under the pointwise operations these germs form is a local ring whose maximal ideal consists of the germs of analytic functions that vanish at p.

(2) (Presheaves) Let $\mathcal{C}_X$ be the category whose objects are the open subsets of X and whose morphisms are the inclusion maps.

(a) A *presheaf* on X is a contravariant functor $F\colon \mathcal{C}_X \to \mathcal{D}$. If $i\colon U \to V$ is an inclusion, we typically write $\rho_{V,U}$ for $F(i)$ and call it *restriction from* V *to* U.
(b) A *morphism* $\eta\colon F \to G$ between presheaves on X is a natural transformation.

(3) (Sheaves)
(a) A *sheaf* on X is a presheaf $F\colon \mathcal{C}_X \to \mathcal{D}$ where $\mathcal{D}$ is a *concrete category*, where F satisfies the *sheaf axioms*.
(i) A category $\mathcal{D}$ is *concrete* if it is equipped with a specified faithful functor $G\colon \mathcal{D} \to \mathbf{SETS}$. (For D and object of $\mathcal{D}$ we may write $d \in D$ when we mean $d \in G(D)$.)
(ii) A functor is *faithful* if it is 1-1 on morphisms.
(iii) The *sheaf axioms* for $F\colon \mathcal{C}_X \to \mathcal{D}$ are:
(A) (Normalization) $F(\emptyset)$ is the terminal object of $\mathcal{D}$ (if it exists). (This is the object in $\mathcal{D}$ which is the codomain of exactly one morphism from each object of $\mathcal{D}$. For equational classes of algebras, like sets, groups, rings, modules, etc., the terminal object is the 1-element object.)
(B) (Local determination) If $s, t \in F(U)$, $\{U_i \mid i \in I\}$ is an open cover of U, and $\rho_{U,U_i}(s) = \rho_{U,U_i}(t)$ for all i, then $s = t$.
(C) (Gluing) If $\{U_i \mid i \in I\}$ is an open cover of U, $s_i \in F(U_i)$ for all $i \in I$, and $\rho_{U_i,U_i\cap U_j}(s_i) = \rho_{U_j,U_i\cap U_j}(s_j)$ for all $i, j \in I$, then there is an $s \in F(U)$ such that $\rho_{U,U_i}(s) = s_i$ for all i.
(b) The *stalk* at p is the colimit (= direct limit) in $\mathcal{D}$ over the cone from F induced by the cone of open sets containing p. (We explained in class what this means, but rather than try to repeat it here I'll just refer you to Chapter 3 of *Categories for the Working Mathematician.*)
(c) A *morphism* $\eta\colon F \to G$ between sheaves on X is just a morphism of presheaves (i.e., a natural transformation).

(4) (B-sheaves) The idea of a B-sheaf over the space X is that of a partially defined sheaf, with $F(U)$ defined only when U is an element from some specified basis for the topology of X. So replace $\mathcal{C}_X$ with a subcategory $\mathcal{B}_X$ whose objects are the open sets from a specified basis for the topology of X and whose morphisms are the inclusion maps.
(a) A B*-sheaf* on X is a presheaf $F\colon \mathcal{B}_X \to \mathcal{D}$, where $\mathcal{D}$ is concrete, satisfying the B-sheaf axioms. These axioms are
(B) (Local determination) If $s, t \in F(U)$, $\{U_i \mid i \in I\}$ is a basic open cover of of the basic open set U, and $\rho_{U,U_i}(s) = \rho_{U,U_i}(t)$ for all i, then $s = t$.
(C) (Gluing) If $\{U_i \mid i \in I\}$ is a basic open cover of the basic open set U, $s_i \in F(U_i)$ for all $i \in I$, and $\rho_{U_i,W}(s_i) = \rho_{U_j,W}(s_j)$ for all $i, j \in I$ and all basic open sets $W \subseteq U_i \cap U_j$, then there is an $s \in F(U)$ such that $\rho_{U,U_i}(s) = s_i$ for all i.

It can be proved that every B-sheaf extends uniquely to a sheaf.

(5) (Affine schemes)

If R is a commutative ring and $X = \mathrm{Spec}(R)$, then there is a B-sheaf of rings, $\mathcal{O}_X$, definable over X, which encodes all the structure of R.

(a) If $\mathfrak{p} \in \mathrm{Spec}(R)$, then $R_\mathfrak{p}$ is local with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. The quotient $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} =: \kappa(\mathfrak{p})$ is called the *residue field* at $\mathfrak{p}$. The composite map $R \to R_\mathfrak{p} \to \kappa(\mathfrak{p})$ assigns to each $f \in R$ its *value* in $\kappa(\mathfrak{p})$.

(b) The value of f at $\mathfrak{p}$ is 0 if the composite map $R \to R_\mathfrak{p} \to \kappa(\mathfrak{p})$ maps f to 0. This means that the canonical map $R \to R_\mathfrak{p}$ maps f into the maximal ideal. This means just that $f \in \mathfrak{p}$. A stronger condition is that f *is zero at* $\mathfrak{p}$: i.e., the canonical map $R \to R_\mathfrak{p}$ maps f to 0 (not just into the maximal ideal). This is equivalent to: $sf = 0$ for some $s \notin \mathfrak{p}$. The set of points where f is zero is open, and the set of points where f has value zero is closed. These sets may be very different; f may be zero nowhere, but have value zero everywhere. (This happens if $\{f\} \cup \mathrm{Ann}_R(f) \subseteq \mathrm{nil}(R)$.)

(c) If $f \in R$, then the *principal open set in Spec*(R) *determined by* f is the set $X_f = D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$. This is the set of points where f has nonzero value. The sets X_f, $f \in R$, form a basis for the topology of $\mathrm{Spec}(R)$, since (i) under the isomorphism from $\mathrm{Ideal}(R)$ to the lattice of open sets the principal ideal (f) maps to X_f, and (ii) the principal ideals $(f) \lhd R$ generate all ideals under join.

(d) We proved in class that $X_g \subseteq X_f$ iff $V(g) \supseteq V(f)$ iff $g \in \sqrt{(f)}$ iff $\exists n, a(g^n = af)$ iff f divides a power of g iff f belongs to the saturation of $\{1, g, g^2, \ldots\}$. This holds iff f maps to a unit under the natural map $R \to R_g$ iff the natural map $R \to R_g$ factors through the natural map $R \to R_f$. Altogether this means that $X_g \subseteq X_f$ holds iff the canonical localization maps induce a map $R_f \to R_g$. Moreover it shows that if $X_f = X_g$, then R_f is canonically isomorphic to R_g. These observations motivate the choices in the definition for the B-sheaf that defines the structure sheaf of $\mathrm{Spec}(R)$.

(e) The *sheaf of regular functions* (or *structure sheaf*) on $\mathrm{Spec}(R)$ is the sheaf of rings determined by the B-sheaf on $\mathrm{Spec}(R)$ defined by $F(X_f) = R_f$ on objects, and for each inclusion $X_g \subseteq X_f$ the corresponding restriction map is defined to be the map $R_f \to R_g$ identified in the preceding item. The stalk at $\mathfrak{p}$ turns out to be the local ring $R_\mathfrak{p}$.

(f) The *affine scheme* associated to R is the pair $(X, \mathcal{O}_X)$ where $X = \mathrm{Spec}(R)$ and $\mathcal{O}_X$ is the sheaf of regular functions on X.

(g) Morphisms between affine schemes were defined in class. I won't repeat the definition here, since it is complicated, but just say that it is possible to equip the class of affine schemes with the appropriate notion of morphism to create a category that is dually equivalent to the category of commutative rings.