Modern Algebra 1 (MATH 6130)
HANDOUT 3 (September 19, 2008)
New material on last page.

CLOSURE SYSTEMS AND CLOSURE OPERATORS

Definition 1. (Closure system) A family $\mathcal{C}$ of subsets of A is a *closure system* if the family is closed under complete intersection. (This includes the empty intersection, which forces $A \in \mathcal{C}$.) The elements of $\mathcal{C}$ are called the *closed* subsets of A with respect to the closure system.

If $X \subseteq A$, then the *closure* of X with respect to this closure system is $\overline{X} := \bigcap\{C \in \mathcal{C} \mid X \subseteq C\}$.

$\overline{X}$ is the smallest closed set containing X.

Example 2. (Of closure systems)

(1) The family of closed sets of a topological space.
(2) The family of subalgebras of an algebra.
(3) The family of down-closed subsets of a partially ordered set $\langle A; \leq \rangle$.

In this course, we are interested in closure systems primarily because of Example (2).

Theorem 3. *The map* $\overline{} : X \mapsto \langle X \rangle$ *is a function* $\mathcal{P}(A) \to \mathcal{P}(A)$ *that is*

(i) *(Extensive)* $X \subseteq \overline{X}$
(ii) *(Isotone)* $X \subseteq Y$ *implies* $\overline{X} \subseteq \overline{Y}$.
(iii) *(Idempotent)* $\overline{\overline{X}} = \overline{X}$.

A function $\overline{} : \mathcal{P}(A) \to \mathcal{P}(A)$ *satisfying (i)—(iii) is called a* closure operator*, and sets of the form* $\overline{X}$ *are called* closed *with respect to* $\overline{}$.

Conversely, given any closure operator $\overline{}$*, the family* $\mathcal{C} = \{\overline{X} \mid X \subseteq A\}$ *is a closure system.*

Proof. Let $\mathcal{C}_X := \bigcap\{C \in \mathcal{C} \mid X \subseteq C\}$. Then $\overline{X} = \bigcap \mathcal{C}_X$, by definition.

Now, for (i), $\overline{X}$ contains X because all members of $\mathcal{C}_X$ contain X. For (ii), if $X \subseteq Y$, then $\mathcal{C}_X \supseteq \mathcal{C}_Y$, so $\overline{X} = \bigcap \mathcal{C}_X \subseteq \bigcap \mathcal{C}_Y = \overline{Y}$. For (iii), we have $X \subseteq \overline{X}$, by (i), so $\overline{X} \subseteq \overline{\overline{X}}$, by (ii). But $\mathcal{C}_X \subseteq \mathcal{C}_{\overline{X}}$ (check), so $\overline{X} = \bigcap \mathcal{C}_X \supseteq \bigcap \mathcal{C}_{\overline{X}} = \overline{\overline{X}}$. Hence $\overline{X} = \overline{\overline{X}}$.

Now suppose that $\overline{}$ is a closure operator. We must show that its closed sets are closed under complete intersection. So, choose any $Z \subseteq \mathcal{P}(A)$ such that every $Y \in Z$ is closed ($Y = \overline{X}$ for some $X \subseteq A$). Let $W = \bigcap Z$. Then since $W = \bigcap Z \subseteq Y$ for any $Y \in Z$ we have $\overline{W} \overset{(ii)}{=} \overline{Y} \overset{(iii)}{=} Y$ for every $Y \in Z$, so $W \overset{(i)}{\subseteq} \overline{W} \subseteq \bigcap Z = W$, yielding $W = \overline{W}$. □

This correspondence between closure systems and closure operators described here is bijective, as one sees by comparing the two senses of "closed set".

Thus the set L of closed sets of a closure operator, when ordered by inclusion, form a complete lattice. (In particular, every subalgebra lattice is complete.) What is less obvious is that, conversely, every complete lattice is isomorphic to the lattice of closed sets of some closure operator.

Theorem 4. *If L is a complete lattice, then L is isomorphic to the lattice of closed subsets of a closure operator.*

Proof. Complete lattices have least and largest elements, 0 and 1. The intervals of the form $[0, a]$, $a \in L$ are closed under arbitrary intersection, since L is complete, with the empty intersection being $[0, 1] = L$. Thus, these intervals form a closure system on the set L. The map $a \mapsto [0, a]$ is a bijection between L and the lattice of closed sets of this closure system which preserves complete meet. Hence it is an isomorphism. $\square$

But not every complete lattice can be realized as a subalgebra lattice. As noticed in class, subalgebra generation has an extra property, listed as item (1) in the next result.

Theorem 5. *If $\overline{} : \mathcal{P}(A) \to \mathcal{P}(A)$ is a closure operator, then the following conditions are equivalent.*

(1) *A subset $X \subseteq A$ that contains the closure of each of its finite subsets is a closed set.*
(2) *The union of an up-directed family of closed sets is closed.*

(A subset of an ordered set $\langle P; \leq \rangle$ is up-directed if whenever $a, b \in P$ there is a $c \in P$ such that $a \leq c$ and $b \leq c$.)

Proof. Suppose that (1) holds and that $\mathcal{D}$ is an updirected family of closed sets. The union $C = \bigcup \mathcal{D}$ contains the closure of any of its finite subsets, as we now prove. If $C_0 \subseteq C$ is finite, then because $\mathcal{D}$ is updirected there is a single $D \in \mathcal{D}$ containing all elements of C_0. This yields $C_0 \subseteq D$, hence $\overline{C_0} \subseteq \overline{D} = D \subseteq C$.

Now suppose that (2) holds, and that $X \subseteq A$ contains the closure of each of its finite subsets. The family $\mathcal{D}$ consisting of the closures of the finite subsets of X is up-directed, so $\bigcup \mathcal{D}$ is closed. But our assumption on X implies that each member of $\mathcal{D}$ is a subset of X, so $\bigcup \mathcal{D} \subseteq X$. On the other hand, $\bigcup \mathcal{D}$ contains every one element subset of X, so $\bigcup \mathcal{D} \supseteq X$. This proves that $X = \bigcup \mathcal{D}$ is closed. $\square$

Corollary 6. *The union of any up-directed family of subalgebras (or congruences) of an algebra is again a subalgebra (or congruence).*

Definition 7. A closure operator is *algebraic* if it satisfies the equivalent conditions of Theorem 5

Algebraic closure operators are special in that one can recognize lattice-theoretically when a closed set is the closure of a finite set.

Definition 8. An element c of a lattice L is *compact* if whenever $c \leq \bigvee Z$ for some $Z \subseteq L$, then there is a finite subset $Z_0 \subseteq Z$ such that $c \leq \bigvee Z_0$.

A complete lattice is *compactly generated* if every element is the least upper bound of a family of compact elements.

A compactly generated complete lattice is called an *algebraic lattice*.

Theorem 9. *The lattice of closed sets of an algebraic closure operator is an algebraic lattice. The compact elements of the lattice are exactly the closures of finite sets.*

Proof. The first statement follows from the second, for the following reason. The lattice of closed sets of any closure operator is complete, according to the remark before Theorem 4. Any closed set c is the union of the closures of its finite sets, since the closure operator is algebraic. If the second statement is true, then the closures of the finite subsets are compact. Thus, every element is the join of compact elements, implying the lattice is algebraic.

To prove the second statement assume that c is the closure of a finite set and that $c \leq \bigvee Z$. Replace $Z \subseteq L$ by the larger set W, whose members are the finite joins of elements of Z ($W = \{\bigvee Z_0 \mid Z_0^{\text{finite}} \subseteq Z\}$). Then W is up-directed and has the same join as Z. This means that $c \subseteq \bigvee Z = \bigvee W = \bigcup W$. The finite set of generators of c must lie in some set $d \in W$, since W is updirected, so $c \leq d$ for some $d = \bigvee Z_0^{\text{finite}} \in W$. This proves that c is compact.

Conversely, assume that c is compact. Let Z be the set of closures of finite subsets of c. Then Z is up-directed and has join equal to $\bigcup Z = c$. By the compactness of c and the up-directedness of Z, $c \in Z$, so c is the closure of a finite set. $\square$

It can be proved that any algebraic lattice is isomorphic to the subalgebra lattice of some algebra and to the congruence lattice of some algebra, so there are no further properties of these lattices to discover.

Applications to algebra.

The remarks above about algebraic closure operators are relevant because of the connection with Zorn's Lemma.

If L is a complete lattice and $c \in L$ is compact, then the ordered set $P := L - [c, 1]$ obtained by deleting everything above c, including c, satisfies the hypotheses of Zorn's Lemma: it is inductively ordered. (This means that any chain in P has an upper bound, namely the join of the chain. This join lies in P because c is compact.) Hence any element of $P = L - [c, 1]$ can be enlarged to a maximal element of $P = L - [c, 1]$.

Rephrasing this in terms of L, *any $x \in L$ that is not above c can be enlarged to an element that is maximal for not being above c.* This fact is useful as a subsitute for induction in settings where induction is not valid.

In particular, if $\mathbf{S}, \mathbf{T} \leq \mathbf{A}$, $\mathbf{T} \not\leq \mathbf{S}$, and $\mathbf{S}$ is finitely generated, then $\mathbf{T}$ can be enlarged to a subalgebra $\mathbf{T}' \leq \mathbf{A}$ that is maximal for the property $\mathbf{S} \not\leq \mathbf{T}'$. A similar statement holds for congruences.

We can describe the same property in terms of elements instead of suabalgebras. If $s \in A$, $\mathbf{T} \leq \mathbf{A}$, and $s \notin T$, then $\mathbf{T}$ can be enlarged to a subalgebra that is maximal for not containing s.

Continuation of Notes

Theorem 10. *The following are equivalent for a complete lattice L.*

(1) *Every element of L is compact.*
(2) *L satisfies the Ascending Chain Condition (ACC).*
(3) *Every nonempty subset of L has a maximal element.*

Proof. $[\neg(2) \Rightarrow \neg(1)]$ The join $c_0 = \bigvee C$ of an infinite properly ascending chain C is not compact, since it is not the join of any finite subchain of C.

$[\neg(3) \Rightarrow \neg(2)]$ If $\emptyset \neq S \subseteq L$ has no maximal element, then one can define an infinite properly ascending chain within S by recursion. (Pick an element in S. It is not maximal, so one can pick a larger element in S. Etc.)

$[(3) \Rightarrow (1)]$ Choose $c \in L$ and a set $Z \subseteq L$ such that $c \leq \bigvee Z$. The set of finite subjoins of elements of Z,

$$\left\{ \bigvee Z_0 \mid Z_0^{\text{finite}} \subseteq Z \right\}$$

is nonempty, so must have a maximal element. This maximal element is a finite join of elements of Z which must equal the join of all elements of Z. Hence there is a finite $Z_0 \subseteq Z$ such that $c \leq \bigvee Z_0$. □

Theorem 11. *In the exact sequence of modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

M is Noetherian iff both L and N are. (Same for Artinian.)

Proof. A modularity argument. □

Corollary 12. *M is Noetherian iff M is finitely generated and all cyclic submodules are Noetherian. (Same for Artinian.)*

Proof. Use induction on the number of generators. For the inductive step, auppose that M is generated by $\{e_1, \ldots, e_n\}$, but by no set of of less than n elements. Apply the theorem to

$$0 \longrightarrow Re_1 \longrightarrow M \longrightarrow M/Re_1 \longrightarrow 0.$$

□

Corollary 13. *A finitely generated module over a Noetherian ring is Noetherian. (Same for Artinian.)*

Proof. Applying Theorem 11 to the exact sequence

$$0 \longrightarrow \mathrm{Ann}(a) \longrightarrow R \longrightarrow Ra \longrightarrow 0$$

yields that a cyclic module Ra over a Noetherian ring R is Noetherian. Thus, when R is Noetherian, Corollary 12 guarantees that any finitely generated module is Noetherian. □

Corollary 13 for Noetherian rings has been called "the Hilbert Basis Theorem for modules".