Rings, Frames, Topological Spaces, Spectra

Commutative Algebra

Sep 9 & 11, 2009

The radical relation

Let R be a commutative ring and let $L = \mathrm{Ideal}(R)$ be its ideal lattice. Define an equivalence relation ρ on L by

$$I \equiv J \pmod{\rho} \iff \sqrt{I} = \sqrt{J}.$$

Facts.

• ρ is compatible with finite meet and arbitrary join. Hence ρ-classes are convex intervals with a top element (= 'teardrop' shape) with the radical ideals being the top elements of each class.

• Therefore the natural map $L \to L/\rho$ is a homomorphism with respect to finite meet and complete join and the quotient lattice L/ρ is a complete lattice.

• L/ρ satisfies the complete distributive law:

$$x \cap \left(\sum y_i\right) = \sum x \cap y_i,$$

making it a *completely distributive lattice*.

Frames

There are different types of morphisms naturally definable for completely distributive lattices, and depending on the choice of morphism one may call these objects 'frames', 'locales', or 'complete Heyting algebras'.

One uses the name 'frame' when one takes as the morphisms the functions $\varphi\colon L \to K$ that preserve finite meet and arbitrary join.

Frames encode topological spaces I: Topological spaces yield frames

If $\langle X; \mathcal{T} \rangle$ is a topological space, then the lattice of open sets under inclusion $\langle \mathcal{T}; \bigcup, \cap \rangle$ is a frame. The lattice of open sets of a topological space is complete: infinitary join is equal to 'set-theoretic union', finitary meet is equal to 'set-theoretic intersection', and infinitary meet is equal to 'interior of set-theoretic intersection'.

Frames encode topological spaces II: Frames yield topological spaces

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Of course, it must be proved that $\mathrm{Spec}(L)$ is a topological space (i.e., the set of sets of the form $D(a)$ is closed under finite intersection and arbitrary union). This is usually done by showing that $V(a) \cup V(b) = V(a \cap b)$ and $\bigcap V(a_i) = V(\sum a_i)$. (The first follows from the distributivity of L and the meet-irreducibility of each point, while the second is trivial.)

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Given a space X we may construct its frame $L(X)$ from which we may construct a space $\mathrm{Spec}(L(X))$. There is a natural function $\varphi\colon X \to \mathrm{Spec}(L(X))$ which maps a point $x \in X$ to the point $p = X \setminus \mathrm{cl}(x) \in L$. φ is continuous. If φ is surjective, then it is a closed mapping, hence φ is a homeomorphism iff it is a bijection.

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• Surjectivity of φ: φ is surjective iff each meet-irreducible open set has the form $X \setminus \mathrm{cl}(x)$ for some $x \in X$, equivalently every join-irreducible closed set is the closure of a point.

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Thm. The category of spatial frames with frame homomorphisms is dually equivalent to the category of sober spaces with continuous maps.

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We have a sequence of constructions:

$$R \rightsquigarrow \mathrm{Ideal}(R) \rightsquigarrow \mathrm{Ideal}(R)/\rho \rightsquigarrow \mathrm{Spec}(\mathrm{Ideal}(R)/\rho)$$

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• Thus $\mathrm{Spec}(R)$ can be constructed on the set of points $P = \{p \in \mathrm{Ideal}(R) \mid p \text{ is prime}\}$ using the topology $\{D(a) \mid a \in \mathrm{Ideal}(R)\}$ where $D(a) = P - V(a)$ and $V(a) = \{b \in P \mid a \leq b\} = V(\sqrt{a})$.

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The facts that (i) all closed sets have the form $V(\sqrt{a})$ and (ii) radical ideals are intersections of the primes that lie above them imply that the frame $\mathrm{Ideal}(R)/\rho$ is spatial. $\mathrm{Spec}(R)$ is sober by construction.

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Proof: (Of the easy direction: $\mathrm{Spec}(R)$ is compact.)
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A system of closed sets $\{V(a_i) \mid i \in I, a_i \in \mathrm{Ideal}(R)\}$ has the FIP iff the set $\{a_i \mid i \in I\}$ has the property that any finite subset generates a proper ideal. One must show that any such set of ideals generates a proper ideal. By contradiction: if $1 \in \langle\{a_i \mid i \in I\}\rangle$, then there would be a finite subset $I_0 \subseteq I$ such that $1 \in \langle\{a_i \mid i \in I_0\}\rangle$.

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Thm. R is directly decomposable iff R has complementary ideals $a, b \lhd R$ iff Spec(R) has complementary closed sets $V(a)$ and $V(b)$ iff Spec(R) is disconnected. Moreover, Spec($R \times S$) is (homeomorphic to) the disjoint union of Spec(R) and Spec(S).

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Exercise. Describe Spec($\mathbb{Z}[x]$).