Exact Sequences, Hom Functors, Tensor Product, II

Commutative Algebra

Sep 21, 2009

Background: Adjoints

Defn. A pair of functors $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ is an *adjoint pair* if there is a natural isomorphism

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• A functor between preadditive categories that has an adjoint must be additive.

• A functor with a left adjoint must be left exact. A functor with a right adjoint must be right exact.

$\otimes$ as a Functor

If $\varphi\colon A \to B$ and $\psi\colon C \to D$ are R-module homomorphisms, then

$$A \times C \xrightarrow{\varphi\times\psi} B \times D \to B \otimes D : (a,c) \mapsto \varphi(a) \otimes \psi(c)$$

is R-bilinear, hence induces an R-module homomorphism

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$\_ \otimes M$ becomes a functor by defining $\varphi \otimes M = \varphi \otimes \mathrm{id}_M$ whenever $A \xrightarrow{\varphi} B$.

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- $\beta \otimes \mathrm{id}$ maps generators onto generators, so it is surjective.
- To show exactness at $M \otimes A$ it suffices to prove that the second map in

$$M \otimes A \xrightarrow{\nu} (M \otimes A)/I \xrightarrow{\overline{\beta\otimes\mathrm{id}}} N \otimes A$$

is invertible, $I = \mathrm{im}(\alpha \otimes \mathrm{id})$. To construct an inverse $\overline{\psi}$, define $\psi\colon N \times A \to (M \otimes A)/I$ by $\psi(n \otimes a) = m \otimes a + I$ for any $m \in \beta^{-1}(n)$. ψ is well-defined because $\ker(\beta) = \mathrm{im}(\alpha)$ and $\mathrm{im}(\alpha) \otimes A = I$. ψ is bilinear, hence extends to a map $\overline{\psi}$ of $N \otimes A$. Since $\overline{\psi} \circ \overline{\beta \otimes \mathrm{id}}$ is the identity on elements $m \otimes a + I$, the map $\overline{\beta \otimes \mathrm{id}}$ is 1-1.

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• Free $\Longrightarrow$ projective $\Longrightarrow$ flat, and none of the arrows can be reversed in general.

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Example. If X and Y are disjoint sets of indeterminates, then $R[X] \otimes_R R[Y] \cong R[X \cup Y]$.