

# Exact Sequences, Hom Functors, Tensor Product, I

## Commutative Algebra

Sep 16 & 18, 2009

# Background: Categories

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Note that if  $A$  is an  $R$ -module, then  $\text{Hom}_{\mathcal{C}}(A, \_)$  may be viewed as a functor from the category of  $R$ -modules to itself. (Addition and scalar multiplication of elements of  $\text{Hom}_{\mathcal{C}}(A, \_)$  are performed pointwise. Same comment for  $\text{Hom}_{\mathcal{C}}(\_, B)$ .)

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$\mathcal{C}$  is *preadditive* if each hom-set  $\text{Hom}_{\mathcal{C}}(A, B)$  has specified abelian group operations such that  $f \circ (g + h) = f \circ g + f \circ h$  and  $(f + g) \circ h = f \circ h + g \circ h$ .

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**Examples.** The category of all  $R$ -modules is preadditive. Representable functors between module categories are additive. The composition of covariant functors represented by  $R$ -modules is additive (and representable): for all  $R$ -modules  $A$  and  $B$

$$\text{Hom}_R(A, \text{Hom}_R(B, X)) \cong \text{Hom}_R(T, X)$$

where  $T = A \otimes_R B$ .

## Definition of $A \otimes_R B$

The best definition of  $A \otimes_R B$  is: it is the module that represents the composite functor  $\text{Hom}_R(A, \text{Hom}_R(B, X))$ . This determines  $A \otimes_R B$  up to isomorphism.

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$A \otimes_R B$  satisfies the universal property associated with this presentation: any function  $f: A \times B \rightarrow M$ , where  $M$  is an  $R$ -module, such that  $f$  preserves the relations will extend to a unique  $R$ -linear map  $\bar{f}: A \otimes_R B \rightarrow M$ . (The  $f$ 's that preserve the relations are exactly the  $R$ -bilinear maps  $A \times B \rightarrow M$ .)

# Chain Complexes

A *chain complex* of  $R$ -modules is a sequence

$$K : \quad \cdots \xrightarrow{\partial_3} K_2 \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\partial_0} K^1 \xrightarrow{\partial^1} K^2 \xrightarrow{\partial^2} \cdots$$

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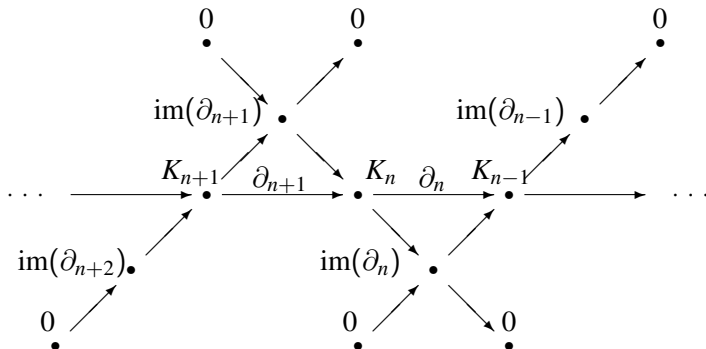
Chain complexes of  $R$ -modules form a category where a morphism  $\alpha: K \rightarrow L$  is an indexed family of  $R$ -linear maps such that all squares are commutative in:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_{n+1}} & K_n & \xrightarrow{\partial_n} & K_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \\ & & \alpha_n \downarrow & & \alpha_{n-1} \downarrow & & \\ \cdots & \xrightarrow{\varepsilon_{n+1}} & L_n & \xrightarrow{\varepsilon_n} & L_{n-1} & \xrightarrow{\varepsilon_{n-1}} & \cdots \end{array}$$

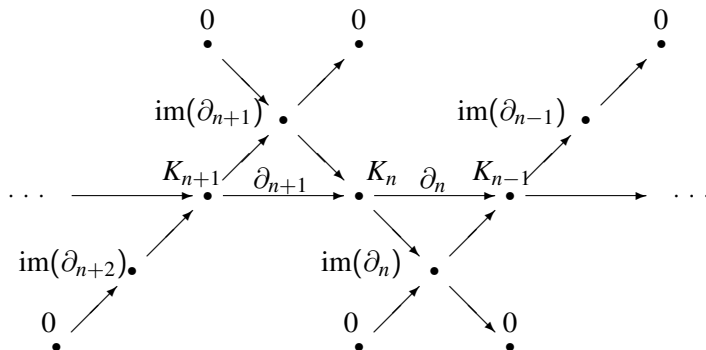
# Factoring a Complex

$$\dots \longrightarrow \overset{K_{n+1}}{\bullet} \xrightarrow{\partial_{n+1}} \bullet \xrightarrow{\overset{K_n}{\partial_n}} \bullet \xrightarrow{K_{n-1}} \bullet \longrightarrow \dots$$

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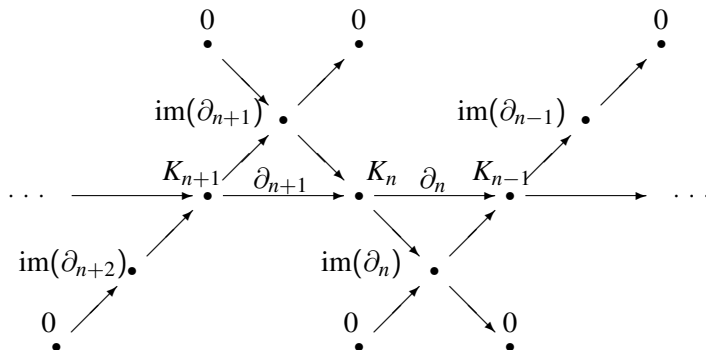


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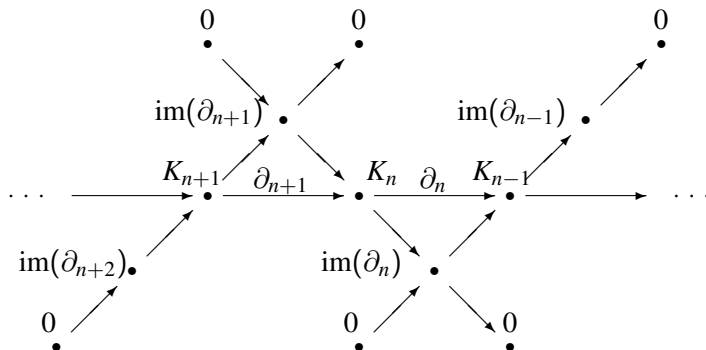
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A contravariant functor is *left exact* if the exactness of  $A \rightarrow B \rightarrow C \rightarrow 0$  implies the exactness of  $0 \rightarrow F(C) \rightarrow F(B) \rightarrow F(A)$ . Etc.

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**Thm.** An  $R$ -module is projective iff it is a retract (= direct summand) of a free  $R$ -module.

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**Thm.** (Baer's Criterion)  $Q$  is an injective  $R$ -module iff the contravariant hom functor it represents preserves exactness of  $0 \longrightarrow I \xrightarrow{\subseteq} R$  for every  $I \triangleleft R$ .