Exact Sequences, Hom Functors, Tensor Product, I

Commutative Algebra

Sep 16 & 18, 2009

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod} \rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod} \rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,

(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod}\rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod} \rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,
(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called the *identity* of A,

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod} \rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,
(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called the *identity* of A,
(5) $\mathrm{dom}, \mathrm{cod} : M \to O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod} \rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,

(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,

(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,

(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called the *identity* of A,

(5) $\mathrm{dom}, \mathrm{cod} : M \to O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod}\rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,
(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called the *identity* of A,
(5) $\mathrm{dom}, \mathrm{cod} : M \to O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

The laws defining categories are:

(1) $f \circ g$ exists if and only if $\mathrm{dom}(f) = \mathrm{cod}(g)$.

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod} \rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,
(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called the *identity* of A,
(5) $\mathrm{dom}, \mathrm{cod} : M \to O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

The laws defining categories are:

(1) $f \circ g$ exists if and only if $\mathrm{dom}(f) = \mathrm{cod}(g)$.
(2) Composition is associative when it is defined.

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod} \rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,
(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called the *identity* of A,
(5) $\mathrm{dom}, \mathrm{cod} : M \to O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

The laws defining categories are:

(1) $f \circ g$ exists if and only if $\mathrm{dom}(f) = \mathrm{cod}(g)$.
(2) Composition is associative when it is defined.
(3) $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$, $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$.

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod}\rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,
(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called the *identity* of A,
(5) $\mathrm{dom}, \mathrm{cod} : M \to O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

The laws defining categories are:

(1) $f \circ g$ exists if and only if $\mathrm{dom}(f) = \mathrm{cod}(g)$.
(2) Composition is associative when it is defined.
(3) $\mathrm{dom}(f \circ g) = \mathrm{dom}(g), \mathrm{cod}(f \circ g) = \mathrm{cod}(f)$.
(4) If $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$, then $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.

Background: Categories

A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod}\rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,
(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called the *identity* of A,
(5) $\mathrm{dom}, \mathrm{cod} : M \to O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

The laws defining categories are:

(1) $f \circ g$ exists if and only if $\mathrm{dom}(f) = \mathrm{cod}(g)$.
(2) Composition is associative when it is defined.
(3) $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$, $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$.
(4) If $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$, then $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.
(5) $\mathrm{dom}(\mathrm{id}_A) = \mathrm{cod}(\mathrm{id}_A) = A$.

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and

(4) $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$.

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and

(4) $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$.

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and

(4) $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism that reverses the directions of all morphisms. Specifically,

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and

(4) $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism that reverses the directions of all morphisms. Specifically,

(1) $F(f \circ g) = F(g) \circ F(f)$,

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and

(4) $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism that reverses the directions of all morphisms. Specifically,

(1) $F(f \circ g) = F(g) \circ F(f)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and

(4) $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism that reverses the directions of all morphisms. Specifically,

(1) $F(f \circ g) = F(g) \circ F(f)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{cod}(F(f))$, and

Background: Functors

A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and

(4) $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism that reverses the directions of all morphisms. Specifically,

(1) $F(f \circ g) = F(g) \circ F(f)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{cod}(F(f))$, and

(4) $F(\mathrm{cod}(f)) = \mathrm{dom}(F(f))$.

Representable Functors

A functor is *representable* if it isomorphic to a hom functor.

Representable Functors

A functor is *representable* if it isomorphic to a hom functor.

That is, a covariant functor is representable if it is naturally isomorphic to a functor of the form $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ for some A. Here $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ maps B to $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and maps φ to "left composition with φ".

Representable Functors

A functor is *representable* if it isomorphic to a hom functor.

That is, a covariant functor is representable if it is naturally isomorphic to a functor of the form $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ for some A. Here $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ maps B to $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and maps φ to "left composition with φ".

A contravariant functor is *representable* if it naturally isomorphic to a functor of the form $\mathrm{Hom}_{\mathcal{C}}(\_, B)$. $\mathrm{Hom}_{\mathcal{C}}(\_, B)$ maps A to $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and maps φ to "right composition with φ".

Representable Functors

A functor is *representable* if it isomorphic to a hom functor.

That is, a covariant functor is representable if it is naturally isomorphic to a functor of the form $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ for some A. Here $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ maps B to $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and maps φ to "left composition with φ".

A contravariant functor is *representable* if it naturally isomorphic to a functor of the form $\mathrm{Hom}_{\mathcal{C}}(\_, B)$. $\mathrm{Hom}_{\mathcal{C}}(\_, B)$ maps A to $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and maps φ to "right composition with φ".

Examples. Um, …

Representable Functors

A functor is *representable* if it isomorphic to a hom functor.

That is, a covariant functor is representable if it is naturally isomorphic to a functor of the form $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ for some A. Here $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ maps B to $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and maps φ to "left composition with φ".

A contravariant functor is *representable* if it naturally isomorphic to a functor of the form $\mathrm{Hom}_{\mathcal{C}}(\_, B)$. $\mathrm{Hom}_{\mathcal{C}}(\_, B)$ maps A to $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and maps φ to "right composition with φ".

Examples. Um, …$\mathrm{Hom}_{\mathcal{C}}(A, \_)$ and $\mathrm{Hom}_{\mathcal{C}}(\_, B)$?

Representable Functors

A functor is *representable* if it isomorphic to a hom functor.

That is, a covariant functor is representable if it is naturally isomorphic to a functor of the form $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ for some A. Here $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ maps B to $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and maps φ to "left composition with φ".

A contravariant functor is *representable* if it naturally isomorphic to a functor of the form $\mathrm{Hom}_{\mathcal{C}}(\_, B)$. $\mathrm{Hom}_{\mathcal{C}}(\_, B)$ maps A to $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and maps φ to "right composition with φ".

Examples. Um, … $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ and $\mathrm{Hom}_{\mathcal{C}}(\_, B)$?

Note that if A is an R-module, then $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ may be viewed as a functor from the category of R-modules to itself. (Addition and scalar multiplication of elements of $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ are performed pointwise. Same comment for $\mathrm{Hom}_{\mathcal{C}}(\_, B)$.)

Background: Additive Functors

$\mathcal{C}$ is *preadditive* if each hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has specified abelian group operations such that $f \circ (g + h) = f \circ g + f \circ h$ and $(f + g) \circ h = f \circ h + g \circ h$.

Background: Additive Functors

$\mathcal{C}$ is *preadditive* if each hom-set $\mathrm{Hom}_{\mathcal{C}}(A,B)$ has specified abelian group operations such that $f \circ (g+h) = f \circ g + f \circ h$ and $(f+g) \circ h = f \circ h + g \circ h$.

A covariant functor $F \colon \mathcal{C} \to \mathcal{D}$ between preadditive categories is *additive* if

$$F \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(F(A),F(B))$$

is an abelian group homomorphism for all $A, B \in \mathrm{Ob}(\mathcal{C})$

Background: Additive Functors

$\mathcal{C}$ is *preadditive* if each hom-set $\mathrm{Hom}_{\mathcal{C}}(A,B)$ has specified abelian group operations such that $f\circ(g+h)=f\circ g+f\circ h$ and $(f+g)\circ h=f\circ h+g\circ h$.

A covariant functor $F\colon\mathcal{C}\to\mathcal{D}$ between preadditive categories is *additive* if

$$F\colon \mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{C}}(F(A),F(B))$$

is an abelian group homomorphism for all $A,B\in\mathrm{Ob}(\mathcal{C})$

Examples. The category of all R-modules is preadditive.

Background: Additive Functors

$\mathcal{C}$ is *preadditive* if each hom-set $\mathrm{Hom}_{\mathcal{C}}(A,B)$ has specified abelian group operations such that $f \circ (g+h) = f \circ g + f \circ h$ and $(f+g) \circ h = f \circ h + g \circ h$.

A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ between preadditive categories is *additive* if

$$F\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(F(A),F(B))$$

is an abelian group homomorphism for all $A,B \in \mathrm{Ob}(\mathcal{C})$

Examples. The category of all R-modules is preadditive. Representable functors between module categories are additive.

Background: Additive Functors

$\mathcal{C}$ is *preadditive* if each hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has specified abelian group operations such that $f \circ (g + h) = f \circ g + f \circ h$ and $(f + g) \circ h = f \circ h + g \circ h$.

A covariant functor $F \colon \mathcal{C} \to \mathcal{D}$ between preadditive categories is *additive* if

$$F \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(F(A), F(B))$$

is an abelian group homomorphism for all $A, B \in \mathrm{Ob}(\mathcal{C})$

Examples. The category of all R-modules is preadditive. Representable functors between module categories are additive. The composition of covariant functors represented by R-modules is additive (and representable): for all R-modules A and B

$$\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, X)) \cong \mathrm{Hom}_R(T, X)$$

where $T = A \otimes_R B$.

Definition of $A \otimes_R B$

The best definition of $A \otimes_R B$ is: it is the module that represents the composite functor $\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, X))$. This determines $A \otimes_R B$ up to isomorphism.

Definition of $A \otimes_R B$

The best definition of $A \otimes_R B$ is: it is the module that represents the composite functor $\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, X))$. This determines $A \otimes_R B$ up to isomorphism.

But one often sees $A \otimes_R B$ defined by a presentation.
$A \otimes_R B = \langle A \times B \mid R \rangle$ where
(i) elements of $A \times B$ are usually written as $a \otimes b$ rather than (a, b).
(ii) R consists of the relations $a \otimes (b + b') = a \otimes b + a \otimes b'$,
$(a + a') \otimes b = a \otimes b + a' \otimes b$, $r(a \otimes b) = ra \otimes b = a \otimes rb$.

Definition of $A \otimes_R B$

The best definition of $A \otimes_R B$ is: it is the module that represents the composite functor $\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, X))$. This determines $A \otimes_R B$ up to isomorphism.

But one often sees $A \otimes_R B$ defined by a presentation.
$A \otimes_R B = \langle A \times B \mid R \rangle$ where
(i) elements of $A \times B$ are usually written as $a \otimes b$ rather than (a, b).
(ii) R consists of the relations $a \otimes (b + b') = a \otimes b + a \otimes b'$,
$(a + a') \otimes b = a \otimes b + a' \otimes b$, $r(a \otimes b) = ra \otimes b = a \otimes rb$.

$A \otimes_R B$ satisfies the universal property associated with this presentation: any function $f : A \times B \to M$, where M is an R-module, such that f preserves the relations will extend to a unique R-linear map $\bar{f} : A \otimes_R B \to M$. (The f's that preserve the relations are exactly the R-bilinear maps $A \times B \to M$.)

Chain Complexes

A *chain complex* of R-modules is a sequence

$$K: \qquad \cdots \xrightarrow{\partial_3} K_2 \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\partial_0} K^1 \xrightarrow{\partial^1} K^2 \xrightarrow{\partial^2} \cdots$$

such that consecutive maps compose to zero. ($\delta_n \circ \delta_{n+1} = 0$)

Chain Complexes

A *chain complex* of R-modules is a sequence

$$K: \qquad \cdots \xrightarrow{\partial_3} K_2 \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\partial_0} K^1 \xrightarrow{\partial^1} K^2 \xrightarrow{\partial^2} \cdots$$

such that consecutive maps compose to zero. ($\delta_n \circ \delta_{n+1} = 0$)

The *nth homology module* of K is $H_n(K) := \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$.

Chain Complexes

A *chain complex* of R-modules is a sequence

$$K: \qquad \cdots \xrightarrow{\partial_3} K_2 \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\partial_0} K^1 \xrightarrow{\partial^1} K^2 \xrightarrow{\partial^2} \cdots$$

such that consecutive maps compose to zero. ($\delta_n \circ \delta_{n+1} = 0$)

The *nth homology module* of K is $H_n(K) := \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$.

K is *exact* if its homology modules are zero.

Chain Complexes

A *chain complex* of R-modules is a sequence

$$K: \qquad \cdots \xrightarrow{\partial_3} K_2 \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\partial_0} K^1 \xrightarrow{\partial^1} K^2 \xrightarrow{\partial^2} \cdots$$

such that consecutive maps compose to zero. ($\delta_n \circ \delta_{n+1} = 0$)

The *nth homology module* of K is $H_n(K) := \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$.

K is *exact* if its homology modules are zero.

Chain complexes of R-modules form a category where a morphism $\alpha\colon K \to L$ is an indexed family of R-linear maps such that all squares are commutative in:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\partial_{n+1}} & K_n & \xrightarrow{\partial_n} & K_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \\
 & & \alpha_n \downarrow & & \alpha_{n-1} \downarrow & & \\
\cdots & \xrightarrow{\varepsilon_{n+1}} & L_n & \xrightarrow{\varepsilon_n} & L_{n-1} & \xrightarrow{\varepsilon_{n-1}} & \cdots
\end{array}$$

Factoring a Complex

$$\cdots \longrightarrow K_{n+1} \xrightarrow{\partial_{n+1}} K_n \xrightarrow{\partial_n} K_{n-1} \longrightarrow \cdots$$

Factoring a Complex

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & \operatorname{im}(\partial_{n+1}) \arrow[ur] \arrow[dr] & & & & \operatorname{im}(\partial_{n-1}) \arrow[ur] & \\
\cdots \arrow[rr] & & K_{n+1} \arrow[ur] \arrow[rr, "\partial_{n+1}"] & & K_n \arrow[rr, "\partial_n"] \arrow[dr] & & K_{n-1} \arrow[ur] \arrow[rr] & & \cdots \\
 & \operatorname{im}(\partial_{n+2}) \arrow[ur] & & & & \operatorname{im}(\partial_n) \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

Factoring a Complex

$$
\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & & & 0 & \\
 & & & & \mathrm{im}(\partial_{n+1}) \arrow[ur] \arrow[dr] & & & & \mathrm{im}(\partial_{n-1}) \arrow[ur] & & \\
\cdots \arrow[rrr] & & & K_{n+1} \arrow[ur] \arrow[rr, "\partial_{n+1}"] & & K_n \arrow[rr, "\partial_n"] \arrow[dr] & & K_{n-1} \arrow[ur] \arrow[rrr] & & & \cdots \\
 & & \mathrm{im}(\partial_{n+2}) \arrow[ur] & & & & \mathrm{im}(\partial_n) \arrow[ur] \arrow[dr] & & & & \\
 & 0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}
$$

$$0 \longrightarrow \mathrm{im}(\partial_{n+1}) \xrightarrow{\subseteq} K_n \xrightarrow{\partial_n} \mathrm{im}(\partial_n) \to 0$$

Factoring a Complex

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & \mathrm{im}(\partial_{n+1}) \arrow[ur] \arrow[dr] & & & & \mathrm{im}(\partial_{n-1}) \arrow[ur] & \\
\cdots \arrow[rr] & & K_{n+1} \arrow[ur] \arrow[rr, "\partial_{n+1}"] & & K_n \arrow[rr, "\partial_n"] \arrow[dr] & & K_{n-1} \arrow[ur] \arrow[rr] & & \cdots \\
 & \mathrm{im}(\partial_{n+2}) \arrow[ur] & & & & \mathrm{im}(\partial_n) \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

$$0 \longrightarrow \mathrm{im}(\partial_{n+1}) \xrightarrow{\subseteq} K_n \xrightarrow{\partial_n} \mathrm{im}(\partial_n) \to 0$$

This short sequence is exact except at the middle, where the homology is the same as in the original sequence.

Factoring a Complex

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & \mathrm{im}(\partial_{n+1}) \arrow[ur] \arrow[dr] & & & & \mathrm{im}(\partial_{n-1}) \arrow[ur] & \\
\cdots \arrow[rr] & & K_{n+1} \arrow[ur] \arrow[rr, "\partial_{n+1}"] & & K_n \arrow[rr, "\partial_n"] \arrow[dr] & & K_{n-1} \arrow[ur] \arrow[rr] & & \cdots \\
 & \mathrm{im}(\partial_{n+2}) \arrow[ur] & & & & \mathrm{im}(\partial_n) \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

$$0 \longrightarrow \mathrm{im}(\partial_{n+1}) \xrightarrow{\subseteq} K_n \xrightarrow{\partial_n} \mathrm{im}(\partial_n) \to 0$$

This short sequence is exact except at the middle, where the homology is the same as in the original sequence. Thus an additive functor preserves exactness of all exact sequences iff it preserves exactness of short exact sequences.

Exactness

F is *left exact* if the exactness of $0 \to A \to B \to C$ implies the exactness of $0 \to F(A) \to F(B) \to F(C)$.

Exactness

F is *left exact* if the exactness of $0 \to A \to B \to C$ implies the exactness of $0 \to F(A) \to F(B) \to F(C)$.

F is *right exact* if the exactness of $A \to B \to C \to 0$ implies the exactness of $F(A) \to F(B) \to F(C) \to 0$.

Exactness

F is *left exact* if the exactness of $0 \to A \to B \to C$ implies the exactness of $0 \to F(A) \to F(B) \to F(C)$.

F is *right exact* if the exactness of $A \to B \to C \to 0$ implies the exactness of $F(A) \to F(B) \to F(C) \to 0$.

F is *exact* if it is left and right exact.

Exactness

F is *left exact* if the exactness of $0 \to A \to B \to C$ implies the exactness of $0 \to F(A) \to F(B) \to F(C)$.

F is *right exact* if the exactness of $A \to B \to C \to 0$ implies the exactness of $F(A) \to F(B) \to F(C) \to 0$.

F is *exact* if it is left and right exact.

A contravariant functor is *left exact* if the exactness of $A \to B \to C \to 0$ implies the exactness of $0 \to F(C) \to F(B) \to F(A)$. Etc.

Left Exactness of Hom

Thm. Hom functors are left exact (whether covariant or not).

Left Exactness of Hom

Thm. Hom functors are left exact (whether covariant or not).

Proof for $\mathrm{Hom}_R(A, \_)$:

Left Exactness of Hom

Thm. Hom functors are left exact (whether covariant or not).

Proof for $\mathrm{Hom}_R(A, \_)$:

Assume $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is exact, and consider

$$0 \longrightarrow H(A,L) \xrightarrow{\alpha\circ} H(A,M) \xrightarrow{\beta\circ} H(A,N).$$

Left Exactness of Hom

Thm. Hom functors are left exact (whether covariant or not).

Proof for $\mathrm{Hom}_R(A, \_)$:

Assume $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is exact, and consider

$$0 \longrightarrow H(A, L) \xrightarrow{\alpha\circ} H(A, M) \xrightarrow{\beta\circ} H(A, N).$$

If $\varphi, \psi \in H(A, L)$ and $\alpha \circ \varphi = \alpha \circ \psi$, then $\alpha(\varphi(a)) = \alpha(\psi(a))$ for every $a \in A$. Since α is 1-1, $\varphi(a) = \psi(a)$ for every $a \in A$, i.e., $\varphi = \psi$.

Left Exactness of Hom

Thm. Hom functors are left exact (whether covariant or not).

Proof for $\mathrm{Hom}_R(A, \_)$:

Assume $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is exact, and consider

$$0 \longrightarrow H(A,L) \xrightarrow{\alpha\circ} H(A,M) \xrightarrow{\beta\circ} H(A,N).$$

If $\varphi, \psi \in H(A,L)$ and $\alpha \circ \varphi = \alpha \circ \psi$, then $\alpha(\varphi(a)) = \alpha(\psi(a))$ for every $a \in A$. Since α is 1-1, $\varphi(a) = \psi(a)$ for every $a \in A$, i.e., $\varphi = \psi$.

If $\varphi \in H(A,M)$ is in the kernel of $\beta \circ \_$ (i.e. $\beta \circ \varphi = 0$), then $\varphi(A) \subseteq \ker(\beta) = \mathrm{im}(\alpha)$. Since α is 1-1, $\alpha^{-1} \circ \varphi \in H(A,L)$ and this is a map which $\alpha \circ \_$ maps to φ. This shows that $\ker(\beta \circ \_) = \mathrm{im}(\alpha \circ \_)$.

Left Exactness of Hom

Thm. Hom functors are left exact (whether covariant or not).

Proof for $\mathrm{Hom}_R(A, \_)$:

Assume $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is exact, and consider

$$0 \longrightarrow H(A, L) \xrightarrow{\alpha\circ} H(A, M) \xrightarrow{\beta\circ} H(A, N).$$

If $\varphi, \psi \in H(A, L)$ and $\alpha \circ \varphi = \alpha \circ \psi$, then $\alpha(\varphi(a)) = \alpha(\psi(a))$ for every $a \in A$. Since α is 1-1, $\varphi(a) = \psi(a)$ for every $a \in A$, i.e., $\varphi = \psi$.

If $\varphi \in H(A, M)$ is in the kernel of $\beta \circ \_$ (i.e. $\beta \circ \varphi = 0$), then $\varphi(A) \subseteq \ker(\beta) = \mathrm{im}(\alpha)$. Since α is 1-1, $\alpha^{-1} \circ \varphi \in H(A, L)$ and this is a map which $\alpha \circ \_$ maps to φ. This shows that $\ker(\beta \circ \_) = \mathrm{im}(\alpha \circ \_)$. $\square$

Nonexactness of Hom, I

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

Nonexactness of Hom, I

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

If you apply $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \_)$ you obtain $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$.

Nonexactness of Hom, I

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

If you apply $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \_)$ you obtain $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$.

Defn. An R-module P is *projective* if it represents an exact covariant hom functor

Nonexactness of Hom, I

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

If you apply $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \_)$ you obtain $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$.

Defn. An R-module P is *projective* if it represents an exact covariant hom functor (equivalently if $\mathrm{Hom}_R(P, \_)$ maps surjections to surjections).

Nonexactness of Hom, I

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

If you apply $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \_)$ you obtain $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$.

Defn. An R-module P is *projective* if it represents an exact covariant hom functor (equivalently if $\mathrm{Hom}_R(P, \_)$ maps surjections to surjections). This means that P is projective if whenever

$$\begin{tikzcd} & P \arrow[d, "\varphi"'] & \\ M \arrow[r, "\beta"] & N \arrow[r] & 0 \end{tikzcd}$$

Nonexactness of Hom, I

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

If you apply $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \_)$ you obtain $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$.

Defn. An R-module P is *projective* if it represents an exact covariant hom functor (equivalently if $\mathrm{Hom}_R(P, \_)$ maps surjections to surjections). This means that P is projective if whenever

$$\begin{tikzcd} & P \arrow[d, "\varphi"'] & \\ M \arrow[r, "\beta"] & N \arrow[r] & 0 \end{tikzcd}$$

there exists $\overline{\varphi}\colon P \to M$ such that $\varphi = \beta \circ \overline{\varphi}$.

Nonexactness of Hom, I

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

If you apply $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \_)$ you obtain $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$.

Defn. An R-module P is *projective* if it represents an exact covariant hom functor (equivalently if $\mathrm{Hom}_R(P, \_)$ maps surjections to surjections). This means that P is projective if whenever

$$\begin{tikzcd} & P \arrow[d, "\varphi"'] & \\ M \arrow[r, "\beta"] & N \arrow[r] & 0 \end{tikzcd}$$

there exists $\overline{\varphi}\colon P \to M$ such that $\varphi = \beta \circ \overline{\varphi}$.
Thm. An R-module is projective iff it is a retract (= direct summand) of a free R-module.

Nonexactness of Hom, II

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

Nonexactness of Hom, II

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

If you apply $\mathrm{Hom}_{\mathbb{Z}}(\_, \mathbb{Z})$ you obtain $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$.

Nonexactness of Hom, II

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

If you apply $\mathrm{Hom}_{\mathbb{Z}}(\_, \mathbb{Z})$ you obtain $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$.

Defn. An R-module Q is *injective* if it represents an exact contravariant hom functor

Nonexactness of Hom, II

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

If you apply $\mathrm{Hom}_{\mathbb{Z}}(\_, \mathbb{Z})$ you obtain $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$.

Defn. An R-module Q is *injective* if it represents an exact contravariant hom functor (equivalently if $\mathrm{Hom}_R(\_, Q)$ maps injections to surjections).

Nonexactness of Hom, II

Example. $0 \longrightarrow \mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q} \xrightarrow{\nu} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ is an exact sequence of $\mathbb{Z}$-modules.

If you apply $\mathrm{Hom}_{\mathbb{Z}}(\_, \mathbb{Z})$ you obtain $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$.

Defn. An R-module Q is *injective* if it represents an exact contravariant hom functor (equivalently if $\mathrm{Hom}_R(\_, Q)$ maps injections to surjections).

Thm. (Baer's Criterion) Q is an injective R-module iff the contravariant hom functor it represents preserves exactness of $0 \longrightarrow I \xrightarrow{\subseteq} R$ for every $I \lhd R$.