DISCRETE MATH
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. What is the definition of the italicized word or phrase? (Hint: the phrase "binary relation" should appear in both answers.)

(a) "F is a ***function*** from A to B".

F is a ***function*** from A to B if F is a binary relation from A to B that satisfies the *function rule*, which is the statement that for all $a \in A$ there is a unique $b \in B$ such that $(a, b) \in F$.

(b) "E is an ***equivalence relation*** on the set A".

E is an ***equivalence relation*** on A if it is a binary relation that is reflexive, symmetric and transitive.

2. Give an example of:

(a) sets $A \in B \in C$ such that $A \nsubseteq B \nsubseteq C$, but $A \subseteq C$.

One example is:
$$\begin{aligned} A &= 1 = \{0\}, \\ B &= \{1\}, \\ C &= \{\{1\}, 0\}. \end{aligned}$$

(b) a function $F\colon \mathbb{R} \to \mathbb{R}$ that is surjective but not injective.

One example is: $F(x) = x^3 - x$. It is not injective since $F(0) = F(1)$, but it is surjective since it is continuous and $\lim_{x\to+\infty} F(x) = +\infty$ and $\lim_{x\to-\infty} F(x) = -\infty$, so F must attain all real values.

3.

(a) What is the recursive definition of multiplication of natural numbers?

$$\begin{array}{lrl} \text{(IC)} & m \cdot 0 & := 0 \\ \text{(RR)} & m \cdot S(n) & := m \cdot n + m \end{array}$$

(b) Prove by induction that $m \cdot 1 = 1 \cdot m = m$ for all $m \in \mathbb{N}$. (You may use any previously proved theorems that concern *addition* only.)

First let's prove that $m \cdot 1 = m$ for all m.

$$\begin{aligned} m \cdot 1 &= m \cdot S(0) && \text{(Defn of 1)} \\ &= m \cdot 0 + m && ((\text{RR}), \cdot) \\ &= 0 + m && ((\text{IC}), \cdot) \\ &= m && \text{(0 is an additive unit, proved earlier)} \end{aligned}$$

Now let's prove by induction the statement "$1 \cdot m = m$".

(Basis of induction, $m = 0$)

$$1 \cdot 0 = 0 \qquad ((\text{IC}), \cdot).$$

(Inductive step) Assume $1 \cdot m = m$ and show $1 \cdot S(m) = S(m)$

$$\begin{aligned} 1 \cdot S(m) &= (1 \cdot m) + 1 && ((\text{RR}), \cdot) \\ &= m + 1 && \text{(Inductive Hypothesis)} \\ &= S(m) && \text{(Theorem "}S(m) = m + 1\text{", from class).} \end{aligned}$$

4.

(a) Let $F\colon \mathbb{R} \to \mathbb{R}$ be a function. Explain in words what property of F is expressed by the sentence "$\exists x\, \forall y\, (F(y) \leq F(x))$".

"There is an x where F attains its absolute maximum value." (In other words, there is an x where the value of F is greater or equal to any other value of F.)

(b) Now suppose that $F(x) = e^x$. Describe a winning strategy for some quantifier which decides the truth of the sentence in (a) in the structure $\langle \mathbb{R}; \leq, F \rangle$.

The sentence is false in the real numbers. A strategy for $\forall$ is: if $\exists$ chooses $x = r$, then $\forall$ chooses $y = r + 1$. Then since $e^{r+1} = F(y) \not\leq F(x) = e^r$ for any choice of r, $\forall$ wins.