DISCRETE MATH (MATH 2001)

REVIEW SHEET II

V. Constructing the integers from the natural numbers
  (a) The natural numbers are well ordered.
  (b) Definition of the integers.
  (c) Definition of the arithmetic and order of the integers.
  (d) Meaning of 'well-defined'. Proofs that the arithmetic operations and order on the integers is well defined.
  (e) Definition of homomorphism. Construction of a 1-1 homomorphism of the natural numbers into the integers.

VI. Counting
  (a) Definitions of cardinality, $|A| = |B|$, $|A| = m$, finite and infinite.
  (b) Proof of the Sum Rule and Product Rule.
  (c) Simple counting formulas:
    (i) There are n^m functions from an m-element set to an n-element set.
    (ii) There are 2^n characteristic functions on an n-element set, and 2^n subsets of an n-element set.
    (iii) There are $n!$ ways to linearly order an n-element set.
    (iv) There are $\binom{n}{k}$ k-elements subsets of an n-element set.
  (d) Binomial and multinomial coefficients: binomial theorem, Pascal's identity, Pascal's triangle. Multinomial theorem, Pascal's identities for multinomial coefficients, Pascal's pyramid.
  (e) Principle of inclusion and exclusion: formula, counting surjective functions, counting derangements.
  (f) Stirling numbers and Bell numbers. Recursion and formula for Stirling numbers. Binomial-type theorem for Stirling numbers.
  (g) Distribution problems.
  (h) Discrete probability: sample space, event, probability distribution, uniform distribution.

VII. Graph theory
  (a) Definitions of: graph, multigraph, directed graph, adjacency, incidence.
  (b) Examples: paths, cycles, complete graphs, complete r-partite graphs, the Petersen graph.
  (c) Planar drawings and planar graphs.
  (d) Euler's Formula: $v - e + r = 2$.
  (e) Characterizations of bipartite graphs.
  (f) Edge bounds for loopless planar graphs with enough vertices ($e \leq 3v - 6$ if $3 \leq v$; $e \leq 2v - 4$ if $4 \leq v$ and the graph is bipartite).
  (g) Kuratowski's characterization of planar graphs.
  (h) Euler characteristic of a compact 2-manifold.

General advice on preparing for a math test.

Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) Describe the procedure for constructing the set of integers from the set of natural numbers. Without proving anything, identify the statement that must be proved to verify that the procedure works.

(2) Show that $m_{\mathbb{Z}} + n_{\mathbb{Z}} = n_{\mathbb{Z}} + m_{\mathbb{Z}}$

(3) How many ways are there to make a circular necklace with n beads of different colors if two necklaces are considered to be the same if they differ by a rotation? What if two necklaces are considered to be the same if they differ by a rotation or a flip?

(4) What is the constant term in $(x^{-2} + 2x^{-1} + 3 + 5x)^3$?

(5) How many loopless multigraphs with vertex set $\{v_1, \ldots, v_n\}$ have k edges? What if loops are allowed?

(6) You have just given birth to octuplets. How many ways can you name your children if you only like the names Billy Bob, Jim Bob and Sue Bob?

(7) If you deal a random 2-card hand, what is the probability of blackjack? (An ace together with a 10 or face card.)

(8) Find the Euler characteristic of the 2-holed torus.

(9) Consider a graph to be a structure $\langle V; E\rangle$ where E is a binary predicate on the set V. Thus $E(a, b)$ holds if vertices a and b are connected by an edge. Write formal sentences that hold in a graph iff
   (a) the graph is loopless.
   (b) any two vertices are connected by a path of length 3.
   (c) K_4 is a subgraph.

(10) Write a formal sentence that distinguishes between the Petersen graph and K_5.

(11) Give a simple description of the class of graphs that satisfy the following sentence.

$$\forall x\ \exists y\ \forall u\ \exists v\ (E(x, y) \wedge E(y, u) \wedge E(u, v) \wedge E(v, x))$$