COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT VII

Read pages 61-92.

PROBLEMS

All rings are commutative.

1. (Lizzi, Martinez) Show that if A is an integrally closed domain and $f \in A[x]$ is a monic polynomial over A, then f irreducible over A iff F is irreducible over the field of fractions of A.

2. (Li, Moore, Tuley) Suppose that $A \leq B$ is an integral extension, and that B is finitely generated as an A-algebra. Show that for every prime $\mathfrak{p} \in \mathrm{Spec}(A)$ there are only finitely many primes of B lying over $\mathfrak{p}$.

3. (Batchelder, Keller, Moorhead) An element $u \in A$ is *integral* over an ideal $I \lhd A$ if u satisfies a monic polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (\dagger)$$

with $a_{n-j} \in I^j$.

- **(a) Show that u is integral over I iff $(I + (u))^n = I(I + (u))^{n-1}$.**
- **(b) Show that u is integral over I iff there is a f.g. A-module M such that $uM \subseteq IM$, and whenever $aM = 0$ then au is nilpotent. (Hint for the forward direction: Start with an equation of integral dependence, ($\dagger$), and let $J \subseteq I$ be a f.g. ideal such that $a_{n-j} \in J^j$ for all j. Let $M = (J + (u))^{n-1}$. Show, using (a), that $uM \subseteq (J + (u))^n = JM \subseteq IM$. Then show that if $aM = 0$ then $(au)^{n-1} = 0$.)**

4. (Jones, Praterelli) The *integral closure* $\overline{I}$ of I is the set of elements of A integral over I.

- **(a) Show that $I \subseteq \overline{I} \subseteq \sqrt{I}$.**
- **(b) Give examples to show that $I \neq \overline{I}$ and $\overline{I} \neq \sqrt{I}$ are possible.**

5. (Hower, Selker) Let A be an integral domain and K its field of fractions. Call $x \in K$ *almost integral* over A if $\exists a \in A \setminus \{0\}$ such that $ax^n \in A$ for all n.

(a) Show that an element of K that is integral over A is almost integral over A.

(b) Show that if A is Noetherian, then any $x \in K$ almost integral over A is integral over A.

(Hint for (b): $A[x]$ is contained in the f.g. A-module $a^{-1}A$.)

6. (Gern, Stanton)

(a) Determine which commutative rings R have the property that the canonical map $R \to R_S$ is surjective for any multiplicatively closed set S. (Hint: Start by considering the case where $S = \{1, f, f^2, \ldots\}$.)

(b) Conclude that every Artinian ring has this property.

(c) Give an example of a non-Artinian ring that has the property.

7. (Chriestensen, Strider) Show that, if $\mathbb{F}$ is a field, then every subring of $\mathbb{F}[x]$ that contains F is Noetherian.