COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT VI

Read pages 54-61 and think about exercises 1-4, 6-7, 10-13

PROBLEMS

All rings are commutative.

1. (Batchelder, Chriestensen) Let $R = \mathbb{F}[x, y]$ **where** $\mathbb{F}$ **is a field. Let** $I = (x^2, xy)$, $P = (x)$ **and** $Q_a = (x^2, y - ax)$, $a \in \mathbb{F}$. **Prove that** $I = P \cap Q_a$ **is a minimal primary decomposition of** I **for any** $a \in \mathbb{F}$. **Prove that** $Q_a \neq Q_b$ **if** $a \neq b$. **Find the associated primes of the primary ideals in the decompositions** $I = P \cap Q_a$ **and identify which are minimal.**

2. (Gern, Hower)

(a) Assume that R **is Noetherian, that** M **is a finitely generated** R**-module and that** $L, N \leq M$ **are submodules. Show that** $L \subseteq N$ **iff** $L_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$ **holds for every** $\mathfrak{p} \in \mathrm{Ass}(M/N)$.

(b) Show that any subset $U \subseteq \mathrm{Spec}(R)$ **can be** $\mathrm{Ass}(M)$ **for some** R**-module** M. **Show that any finite subset** $U_0 \subseteq \mathrm{Spec}(R)$ **can be the set of associated primes of some f.g. module.**

3. (Jones, Keller, Li) Throughout this problem R **is Noetherian,** M **is a f.g.** R**-module and** N **is an arbitrary** R**-module.**

(a) Assume in addition that R **is local with maximal ideal** $\mathfrak{m}$. **Show that** $\mathfrak{m} \in \mathrm{Ass}(\mathrm{Hom}_R(M, N))$ **iff** $M \neq 0$ **and** $\mathfrak{m} \in \mathrm{Ass}(N)$. **(Hint: to go from left to right consider whether** $\mathrm{Hom}(R/\mathfrak{m}, \mathrm{Hom}(M, N))$ **is nontrivial.)**

(b) Now drop the assumption that R **is local. Use part (a) and localization to prove that**

$$\mathrm{Ass}(\mathrm{Hom}_R(M, N)) = \mathrm{Supp}(M) \cap \mathrm{Ass}(N).$$

You may use the fact that the map

$$(\mathrm{Hom}_R(M, N))_S \to \mathrm{Hom}_{R_S}(M_S, N_S) \colon \frac{\varphi}{s} \mapsto \left(\frac{m}{t} \mapsto \frac{\varphi(m)}{st} \right)$$

is an isomorphism of R_S**-modules whenever** R **is Noetherian and** M **is f.g. as an** R**-module (Lemma 11.32 of Rotman's *Advanced Modern Algebra*).**

4. (Lizzi, Martinez)

(a) **Prove that if** $0 \to L \to M \to N \to 0$ **is exact, then** $\mathrm{Supp}(M) = \mathrm{Supp}(L) \cup \mathrm{Supp}(N)$**.**
(b) **Prove that** $\mathrm{Supp}(L \otimes N) = \mathrm{Supp}(L) \cap \mathrm{Supp}(N)$**.**
(c) **(Unrelated to (a) and (b).) Show that if** R **is Noetherian, then the total ring of fractions[1] of** R **has finitely many maximal ideals. (I.e.,** R_S **is *semilocal*.)**

A ring is *subdirectly irreducible* (SI) if it has a least nonzero ideal. A module is SI if it has a least nonzero submodule.

5. (Moore, Moorhead) Prove in each of the following ways that an SI Noetherian ring must be Artinian:

(a) **Using associated primes: Show that** R **has a unique associated prime, which is a nilpotent maximal ideal. Then show that a Noetherian ring with a nilpotent maximal ideal is Artinian.**
(b) **Using the Krull Intersection Theorem: again, first show that** R **has a nilpotent maximal ideal.**

6. (Praterelli, Selker) Let M **be a f.g. SI module over a Noetherian ring** R**.**

(a) **Show that** M **is Artinian.**
(b) **Show that** M **has a composition series, and that all composition factors are isomorphic.**

7. (Stanton, Strider, Tuley) Let R **be a infinite Noetherian integral domain of cardinality** ρ **that has a maximal ideal of index** κ**.**

(a) **Use the Krull Intersection Theorem to prove that** $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$**.**
(b) **Show that if** ρ **and** κ **are infinite cardinals satisfying** $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$**, then there is a Noetherian integral domain** R **of cardinality** ρ **with a maximal ideal of index** κ**. (Hint: consider rings** R **such that** $\mathbb{F}[x] \subseteq R \subseteq \mathbb{F}[[x]]$ **where** $\mathbb{F}$ **is a field of cardinality** κ**.)**
(c) **Give examples to show that it is not possible to bound** ρ **in terms of** κ **if we drop either the hypothesis that** R **is Noetherian or the hypothesis that** R **is an integral domain.**

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[1]The *total ring of fractions* of R is the ring R_S where $S \subseteq R$ is the set of elements that are not zero divisors. This is the greatest localization for which the canonical map $R \to R_S$ is an embedding.