COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT V

Read pages 45-54.

PROBLEMS

All rings are commutative.

1. (Gern, Jones) Let R be a PID and K be its field of fractions.

(a) Show that any intermediate subring $R \subseteq S \subseteq K$ is a localization of R, and is also a PID.

(b) Suppose now that R is also a local ring. Show that the only intermediate rings $R \subseteq S \subseteq K$ are $S = R$ and $S = K$.

(c) Show that parts (a) and (b) are false for UFD's in place of PID's.

2. (Li, Strider) Show that the following are equivalent.

(a) $R_{\mathfrak{p}}$ is a field for every prime ideal $\mathfrak{p} \lhd R$.

(b) $R_{\mathfrak{m}}$ is a field for every maximal ideal $\mathfrak{m} \lhd R$.

(c) R is a regular ring. (R is *regular* if every finitely generated ideal of R is idempotent. Such rings are also called *absolutely flat*, since these are the rings whose modules are all flat.) Deduce that if R is an integral domain, then (a) and (b) are equivalent to (c)': R is a field.

3. (Moore, Selker) Let $N \leq M$ be R-modules, $I \lhd R$ an ideal of R, and $m, n \in M$ and $r \in R$ be elements.

(a) Show that the set of primes $\mathfrak{p}$ where "$m \in N_{\mathfrak{p}}$" is an open subset of $\mathrm{Spec}(R)$, and that it is all of $\mathrm{Spec}(R)$ iff $m \in N$. (Here $m \in N_{\mathfrak{p}}$ is shorthand for $\frac{m}{1} \in N_{\mathfrak{p}}$.)

(b) Show that the set of primes $\mathfrak{p}$ where "$m = 0$" is an open subset of $\mathrm{Spec}(R)$, and that it is all of $\mathrm{Spec}(R)$ iff $m = 0$.

(c) Same type of problem for "$m = n$".

(d) Same type of problem for "r is nilpotent".

(e) Same type of problem for "r is a unit".

4. (Batchelder, Keller) Let $L, N \leq M$ be R-modules. Let U be the set of primes for which $L_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$ holds. Show that U is an intersection of open sets in $\mathrm{Spec}(R)$. Show conversely that if V is any intersection of open sets in $\mathrm{Spec}(R)$, then V is exactly the set of primes for which $L_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$ holds for some submodules L, N of some module M.

5. (Lizzi, Moorhead) Show that the weak form of Nakayma's Lemma is equivalent to the statement: (†) "If R is a local ring and M and N are finitely generated R-modules, then $M \otimes_R N = 0$ implies $M = 0$ or $N = 0$." Here "equivalent" means in some vague way that each statement can be derived from the other without using the ideas from the proof of Nakayma's Lemma. (Hint: For one direction ca3p6(a) is relevant. For the other direction, imagine how one might try to prove Nakayama's Lemma by a localization argument.)

6. (Martinez, Tuley) A topological space is *Noetherian* if it satisfies the ascending chain condition on open sets.

(a) Show that a space is Noetherian iff every open subspace is compact iff every subspace is compact.
(b) Show that if R is a Noetherian ring, then $\mathrm{Spec}(R)$ is Noetherian.
(c) Give an example of a non-Noetherian ring R where $\mathrm{Spec}(R)$ is a Noetherian topological space.

7. (Praterelli, Stanton)

(a) Show that a Noetherian space is a union of finitely many irreducible closed sets. ("Irreducible" in this sense means $\cup$-irreducible: $A = B \cup C$ implies $A = B$ or $A = C$.)
(b) Using part (a), give a topological proof of the fact that if R is Noetherian and $I \lhd R$ is an ideal, then there are finitely many minimal primes over I.

8. (Chriestensen, Hower)

(a) Show that R is Noetherian iff $\mathrm{Spec}(R)$ has a cover of principal open sets $\{X_f\}$ such that each R_f is a Noetherian ring. (Hint: Show that any $I \lhd R$ is the intersection of the ideals obtained from I by extending and contracting along each of the maps $R \to R_f$.)
(b) Is it true that R is Noetherian iff $R_{\mathfrak{p}}$ is Noetherian for all $\mathfrak{p}$?