COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT IV

Read the pages 32-43. Note that the first 2 exercises of Chapter 3 give a new proof of the strong form of Nakayama's Lemma using localization.

PROBLEMS

All rings are commutative.

1. (Wane) Show that if $I, J \lhd R$, then $R/I \otimes_R R/J \cong R/(I+J)$.

2. (Lizzi, Moore) Let A and B be R-algebras, and define $i_A\colon A \to A \otimes_R B\colon a \mapsto a \otimes 1$ and $i_B\colon B \to A \otimes_R B\colon b \mapsto 1 \otimes b$.

(a) Show that $(A \otimes B, i_A, i_B)$ is a coproduct of A and B in the category of R-algebras.

(b) Show that if A is the R-algebra presented by generators and relations as $A = \langle G \mid S \rangle$ and $B = \langle H \mid T \rangle$ (with $G \cap H = \emptyset$), then $A \otimes_R B \cong \langle G \cup H \mid S \cup T \rangle$.

3. (Keller, Selker) Let $\mathbb{F}$ be a field. Suppose that A and B are $\mathbb{F}$-algebras and that $B = \mathbb{F}[b]$ is generated as an $\mathbb{F}$-algebra by a single element $b \in B$.

(a) Show that $A \otimes_{\mathbb{F}} B \cong A[x]/\min_{b,\mathbb{F}}(x)$.

(b) Restrict now to the case where A and B are fields. Give an example where $A \otimes_{\mathbb{F}} B$ has nonzero nilpotent elements, and another example where $A \otimes_{\mathbb{F}} B$ ($\neq 0$) has no nonzero nilpotent elements.

4. (Batchelder, Jones, Tuley) Let m be an integer that is not a perfect square.

(a) Show that $\mathbb{Q}[\sqrt{m}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{m}] \cong \mathbb{Q}[\sqrt{m}] \times \mathbb{Q}[\sqrt{m}]$ as $\mathbb{Q}$-algebras.

(b) Find the idempotents in $\mathbb{Q}[\sqrt{m}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{m}]$ that induce the direct decomposition descibed in (a).

(c) Find an idempotent $e \neq 0, 1$ in $\mathbb{Q}[\sqrt[3]{2}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt[3]{2}]$.

5. (Gern, Stanton) (There is no contravariant version of tensor product.)

(a) Show that the composite of two contravariant representable functors from the category of R-modules to itself is an additive covariant functor.

(b) Show that the composite of two contravariant representable functors from the category of R-modules to itself is not representable when R is a field.

6. (Hower, Martinez) Show that if A is a flat R-module, then its character module $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is an injective R-module. (The converse is true, but you don't need to prove it.)

7. (Chriestensen, Strider) Suppose that

$$0 \longrightarrow \mathrm{Hom}_R(C, M) \xrightarrow{\circ\psi} \mathrm{Hom}_R(B, M) \xrightarrow{\circ\varphi} \mathrm{Hom}_R(A, M)$$

is exact for every R-module M. Show that

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

is exact. (Hint: Consider what happens when $M = C/\mathrm{im}(\psi)$, $M = C$ and $M = B/\mathrm{im}(\varphi)$.)

8. (Li, Moorhead, Praterelli) Show that if M is free, projective, flat, finitely generated, or finitely presentable as an R-module, then so is M_S as an R_S-module.