COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT III

Read the pages 21-31.

PROBLEMS

All rings are commutative.

1. (Li, Selker, Stanton) Here we consider Spec(R) as a topological space (the primes equipped with the Zariski topology), and as an ordered set (the primes equipped with the inclusion order).

(a) Show that the inclusion order on the prime ideals can be recovered from the topology of Spec(R).

(b) Show that conversely, if R is a Noetherian ring, then the topology of Spec(R) can be determined from the inclusion order on the prime ideals.

(c) Show that if R is not Noetherian, then the topology of Spec(R) may not be recoverable from the inclusion order on the primes.

2. (Praterelli, Strider, Tuley)

(a) Suppose that R is a UFD. Show that a prime ideal in R is generated by some set of irreducible elements.

(b) Now suppose that $R = S[x]$ where S is a PID. Show that any prime ideal of R is generated by at most 2 irreducible elements. Show that if a prime ideal requires two irreducible generators, then it has the form $I = (p, f(x))$ where p is prime in S and $f(x)$ is a monic polynomial in $S[x]$ that is irreducible mod p.

(c) (continued from (b)) Sketch the ordered set of primes of $S[x]$ under inclusion to the best of your ability. How long can a chain be?

3. (Gern, Hower)

(a) Suppose that a and b are elements of a modular lattice. Show that the the canonical isomorphisms

$$\begin{aligned} [a \cap b, a] &\to [b, a+b] \quad &: x \mapsto x + b \\ [b, a+b] &\to [a \cap b, a] \quad &: y \mapsto y \cap a \end{aligned}$$

map compact elements to compact elements. Deduce that M is finitely generated over $M \cap N$ iff $M + N$ if finitely generated over N.

(b) Prove that if $M \cap N$ and $M + N$ are both finitely generated, then so are M and N.

4. (Jones, Moore)

(a) Let M be a finitely generated faithful R-module. Show that M is a Noetherian module iff R is a Noetherian ring.

(b) Suppose that R is a subring of S and that S is finitely generated as R-module. (In this situation we say that S is ***finite over*** R, or S is a ***finite*** R-algebra.) Show that R is Noetherian iff S is Noetherian.

5. (Lizzi, Martinez, Wane) Define the (Jacobson) radical of an R-module M to be

$$\mathbf{rad}(M) = \bigcap_{N \prec M} N.$$

(a) Prove that $\mathbf{rad}(R)M \subseteq \mathbf{rad}(M)$.

(b) Show that $\mathbf{rad}(M)$ consists of the ***nongenerators*** of M: i.e., $m \in \mathbf{rad}(M)$ iff $M = \langle S \cup \{m\} \rangle$ implies $M = \langle S \rangle$. (This means single elements of $\mathbf{rad}(M)$ may be cancelled from any generating set.)

(c) Show that if M is finitely generated and $P \subseteq \mathbf{rad}(M)$, then $M = N + P$ implies $M = N$. (This means any set of elements of $\mathbf{rad}(M)$ may be cancelled from a generating set of a finitely generated module.) In particular, show that if $I \subseteq \mathbf{rad}(R)$, M is finitely generated, and $M = N + IM$, then $M = N$. (This statement is a variant of the weak form of Nakayama's Lemma.)

(d) Prove that if a nonzero module M is finitely generated, then $\mathbf{rad}(M)$ is a proper submodule of M. Deduce the weak form of Nakayama's Lemma from this.

6. (Chriestenson, Keller) Suppose that $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$, and that M is a finitely generated R-module.

(a) Show that a subset $F \subseteq M$ is a generating set iff $F/\mathfrak{m}$ is a generating set for the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$. Conclude that all minimal generating sets for M have the same size.

(b) Show that a homomorphism of $\varphi \colon M \to N$ between finitely generated R-modules is surjective iff the induced map $\varphi_{\mathfrak{m}} \colon M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective. (Your solution should say why there is an "induced map".)

7. (Batchelder, Moorhead) An algebra is said to be ***Hopfian*** if every surjective endomorphism is an automorphism.

(a) Prove that every finitely generated module over a commutative ring is Hopfian. (Hint: Suppose that M is a finitely generated R-module and that $\varphi\colon M \to M$ is surjective. Apply the strong form of Nakayama's Lemma to M considered as an $R[\varphi]$-module.)

(b) Give an example of a finitely generated module over a noncommutative ring that is not Hopfian. Explain why no such module can be Noetherian.