COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT II

Read the pages 17-20.

PROBLEMS

All rings are commutative.

1. (Hill, Jones)

(a) Show that nil$(R \times S) =$ nil$(R) \times$nil(S) and rad$(R \times S) =$ rad$(R) \times$rad(S). Hence, if the nilradical and the Jacobson radical are equal in each coordinate of a product, then they are equal in the product.

(b) Show the result of part (a) does not hold for infinite products by showing that the nilradical and Jacobson radical are equal in all coordinates of the product $T = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \cdots$, but nil$(T) \neq$ rad(T).

2. (Li, Martinez, Moore) Prove that the Jacobson radical contains no nonzero idempotents in each of the following ways:

(a) using the characterization of rad(R) as the intersection of maximal ideals.

(b) using the characterization of rad(R) as the largest ideal J such that $1 + J$ consists of units.

(c) using the characterization of rad(R) as the intersection of annihilators of all simple modules.

3. (Praterelli, Selker) Show that rad(R) and nil(R) can be characterized in the following ways.

(a) rad(R) is the largest ideal $J \lhd R$ such that all covers below J in Ideal(R) are of abelian type. (That is, $I \prec K \leq J$ implies $K^2 \subseteq I$.)

(b) nil(R) is the largest ideal $I \lhd R$ such that there is a well-ordered chain of ideals

$$0 = I_0 \leq I_1 \leq I_2 \leq \cdots \leq I_\mu = I$$

such that

(i) $I_{\alpha+1}$ is abelian over I_α for all α, and

(ii) $I_\lambda = \bigcup_{\kappa<\lambda} I_\kappa$ whenever λ is a limit ordinal.

4. (Stanton, Batchelder, Hower) Prove that

(a) the collection of radical ideals ordered by inclusion is a complete lattice.

(b) the lattice of radical ideals satisfies the complete distributive law,

$$x \cap \sum y_i = \sum x \cap y_i.$$

(c) the function $\sqrt{\ } : I \mapsto \sqrt{I}$ is a surjective $+$-complete homomorphism from Ideal(R) to the lattice of radical ideals.

5. (Keller, Lizzi) Prove that the intersection of a chain of prime ideals is a prime ideal. Conclude that every ideal I has minimal prime ideals that contain it, and that $\sqrt{I}$ is the intersection of these minimal primes.

6. (Strider, Moorhead, Gern) Suppose that $I \lhd R$ has infinitely many minimal primes that contain it.

(a) Show that I is not prime.

(b) Use (a) to show that there is an ideal properly containing I that also has infinitely many minimal primes above it.

(c) Conclude that R is not Noetherian. (Expressed more positively, any Noetherian ring has the property that every ideal I has only finitely many minimal primes containing it, hence $\sqrt{I}$ is an intersection of finitely many primes.)

7. (Chriestenson, Tuley, Wane) Show that

(a) the only idempotents of a local ring are 0 and 1.

(b) there are nonlocal rings whose only idempotents are 0 and 1.

(c) if R is Artinian and its only idempotents are 0 and 1, then R is local. Conclude that an Artinian ring is a finite product of local rings.