COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT I

Read Chapter 1

PROBLEMS

All rings are commutative.

1. (Batchelder, Chriestenson, Gern) Let k be a field. Describe the ideal lattices of

(a) $k[x]$.
(b) $k[[x]]$ **(= the ring of formal power series over k, $f = \sum_{n=0}^{\infty} a_n x^n$).**
(c) $k((x))$ **(= the ring of Laurent series over k, $f = \sum_{n=r}^{\infty} a_n x^n$, $r \in \mathbb{Z}$).**

2. (Hill, Hower) Let R be an integral domain and let K be its field of fractions. Show that the following are equivalent.

(a) For every $x \in K$, either $x \in R$ or $x^{-1} \in R$.
(b) The ideal lattice of R is a chain.

3. (Jones, Keighley, Keller, Lage)

(a) Show that a ring R is directly decomposable as a ring iff it is directly decomposable when considered as an R-module.
(b) Show that an R-module M is directly decomposable iff it has an idempotent endomorphism $\varepsilon\colon M \to M$ such that $\ker(\varepsilon) \neq 0 \neq \mathrm{im}(\varepsilon)$.
(c) Show that the R-module endomorphisms of $_RR$ all have the form $\varepsilon(x) = rx$ for some $r \in R$.
(d) Show that any direct decomposition of R has the form $R \cong R/(e) \times R/(1-e)$ for some idempotent $e \in R$.

4. (Li, Lizzi) Show that the ideals of $R \times S$ are of the form $I \times J$ where $I \lhd R$ and $J \lhd S$. Show that the prime (maximal) ideals have the form $P \times S$ and $R \times Q$ for prime (maximal) ideals $P \lhd R$ and $Q \lhd S$.

5. (Martinez, Strider) Let $\varphi\colon R \to S$ be a surjective homomorphism of rings.

(a) Show that $\varphi(\mathbf{nil}(R)) \subseteq \mathbf{nil}(S)$ and $\varphi(\mathbf{rad}(R)) \subseteq \mathbf{rad}(S)$ with equality if $\ker(\varphi)$ is nil.

(b) Give examples to show that equality need not hold in (a) if $\ker(\varphi)$ is not nil.

6. (Moore, Moorhead) Suppose that $I \lhd R$ is nil.

(a) Show that $a + I$ is a unit in R/I iff a is a unit in R.

(b) Show that $a + I$ is idempotent in R/I iff there exists an idempotent $e \in R$ such that $e + I = a + I$. (Idempotents can be lifted modulo a nil ideal.) (Hint for "only if": use the fact that $[a(1-a)]^n = 0$ for some n, then expand $(a + (1-a))^{2n}$.)

7. (Praterelli, Roy) Suppose that $I \lhd R$ is finitely generated. Show that $I^2 = I$ iff $I = (e)$ for some idempotent element e. Give an example of a ring with a nil ideal I satisfying $I^2 = I$.

8. (Selker, Wane) Show that rad(R) is the largest ideal J such that $1 + J$ consists of units.

9. (Stanton, Tuley) Show that if R is a commutative ring, then R is a homomorphic image of a subring of a field. Conclude that commutative rings satisfy all positive universal[1] sentences true in all fields. Explain how this shows that (for example) the truth of the Cayley-Hamilton Theorem for fields implies the truth of this theorem for any commutative ring.

[1] A sentence, "$Q_1x_1 \cdots Q_nx_n$(quantifier-free part)", in an algebraic language, where the Q's are quantifiers, is *positive* if the quantifier-free part is built up from equations using only "and" and "or" and is *universal* if the quantifiers are all universal quantifiers ($\forall$).