Associated Primes For Modules

Commutative Algebra

Oct 14-21, 2009

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• Cyclic modules of the form $R/\mathfrak{p}$ behave like the more specialized modules $S = R/\mathfrak{m}$ ($\mathfrak{m}$ maximal), which are typical simple modules. It is not unreasonable to think of associated primes as specifying a kind of torsion in M: $\mathfrak{p} \in \mathrm{Ass}(M)$ iff $\mathfrak{p} = (0 : m)$ for some $m \in M$, so m is an "exact $\mathfrak{p}$-torsion" element.

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Proof. (1) Assume that $(0 : m)$ is maximal in $\mathcal{S}$, $rs \in (0 : m)$, but $s \notin (0 : m)$. Then $rs \in (0 : m)$ iff $rsm = 0$ iff $(0 : sm) \supseteq \{r\} \cup (0 : m)$. Since $sm \neq 0$, the maximality of $(0 : m)$ in $\mathcal{S}$ implies that $\{r\} \cup (0 : m) = (0 : m)$, or $r \in (0 : m)$.

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(2b) The set of zero divisors on M is $\bigcup_{m \in M \setminus \{0\}} (0 : m)$, which by (2a) is the union $\bigcup \mathrm{Ass}(M)$ of the maximal ideals in $\mathcal{S}$.

Exact Sequences

Thm. If $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is exact, then

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Example. If $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}_2 \times \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}_2 \times \mathbb{Z}_3 \to 0$ is exact, where $\alpha(m) = (0, 3m)$, then $\mathrm{Ass}(L) = \{(0)\}$, $\mathrm{Ass}(M) = \{(0), (2)\}$, and $\mathrm{Ass}(N) = \{(2), (3)\}$. Thus none of the inclusions need be equality.

$\mathrm{Ass}(M)$ is a Finite Set

Thm. If M is a nonzero f.g. module over a Noetherian ring R, then M has a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

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Proof. $\mathrm{Ass}(M) \neq \emptyset$, so $\exists M_1 \leq M$ such that $M_1 \cong R/\mathfrak{p}_1$. Repeat with M/M_1 to obtain a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots$ of the desired type, which must be finite since M is Noetherian.

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Sketch of proof. The theorem on exact sequences implies that every associated prime of M must arise as a factor in any filtration of M with factors of the form $R/\mathfrak{p}$. Now apply the above theorem.

Associated Primes Under Localization, I

Here we consider the effect on associated primes of restriction of scalars along $R \to R_S$. ($R \to R_S$ induces a continuous injection $\mathrm{Spec}(R_S) \to \mathrm{Spec}(R)$, which we treat as inclusion.)

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Proof. If $m \in M$, then $(0 : m)_R = (0 : m)_{R_S} \cap R$. Hence if $\mathfrak{p} \in \mathrm{Ass}_{R_s}(M)$ we have $\mathfrak{p}|_R \in Ass(M)$.

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Conversely if $m \in M \setminus \{0\}$ and $\mathfrak{p} = (0 : m)_R \in \mathrm{Ass}_R(M)$, then $\mathfrak{p} \cap S = \emptyset$ (else $m = 0$), so $\mathfrak{p}R_s$ is prime. We claim that $\mathfrak{p}R_s = (0 : m)_{R_S}$, so that $\mathfrak{p}R_s \in \mathrm{Ass}_{R_S}(M)$. To see this, note that $r/s \in (0 : m)_{R_S}$ iff $\exists t \in S(trm = 0)$ iff $\exists t \in S(tr \in (0 : m)_R = \mathfrak{p})$ iff $r \in \mathfrak{p}$.

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Proof. If $\mathfrak{p} \in \mathrm{Ass}_R(M) \cap \mathrm{Spec}(R_S)$, then $\mathfrak{p} = (0 : m)_R$ for some $m \in M \setminus \{0\}$ and $\mathfrak{p}$ is disjoint from S.

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• If $\mathfrak{p} \in \mathrm{Supp}(M)$, then there is a prime $\mathfrak{p}' \subseteq \mathfrak{p}$ such that $\mathfrak{p}'$ is minimal in $\mathrm{Supp}(M)$: by HWV(3b), the set $\mathrm{Supp}(m)$ (= primes where $m \neq 0$) is closed. If M is generated by $m_1, \ldots, m_k$, then $\mathrm{Supp}(M) = \bigcup_{i=1}^k \mathrm{Supp}(m_i)$ is closed, hence $\mathrm{Supp}(M) = V(I)$ for some ideal I. But any prime $\mathfrak{p} \supseteq I$ contains a minimal prime $\mathfrak{p} \supseteq \mathfrak{p}' \supseteq I$ by HWII(5).

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• Any minimal $\mathfrak{p} \in \mathrm{Supp}(M)$ is in $\mathrm{Ass}(M)$: $\mathfrak{p} \in \mathrm{Supp}(M)$, so $M_\mathfrak{p} \neq (0)$, so

$$\emptyset \neq \mathrm{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{Ass}_R(M) \cap \mathrm{Spec}(R_\mathfrak{p}) \subseteq \mathrm{Supp}(M) \cap \mathrm{Spec}(R_\mathfrak{p}) = \{\mathfrak{p}\}.$$

Primary Submodules

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- the set $\mathfrak{p}$ of $r \in R$ where λ_r is not injective is a prime ideal. (N is $\mathfrak{p}$-*primary*.)

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Defn. A submodule $N \lneq M$ is *primary* if for all $r \in R$ the map $\lambda_r : M/N \to M/N : \overline{m} \to r\overline{m}$ is injective or nilpotent.

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- If $N \lneq M$ is $\mathfrak{p}$-primary and $m \in M \setminus N$, then $(N : m)$ is a $\mathfrak{p}$-primary ideal.

Connection Between Associated Primes and Primary Submodules, I

Assume that R is Noetherian and M is a f.g. R-module.

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Primary Decomposition For Modules

Thm. If $Q \lneq M$ is a meet-irreducible R-submodule, then $\mathrm{Ass}(M/Q)$ has at most one element. If also R is Noetherian and M is f.g., then $\mathrm{Ass}(M/Q)$ has exactly one element, and Q is primary.

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Thm. If $N = Q_1 \cap \cdots \cap Q_k$ then $\mathrm{Ass}(M/N) \subseteq \bigcup \mathrm{Ass}(M/Q_i)$. When R is Noetherian, equality holds provided $N = Q_1 \cap \cdots \cap Q_k$ is an irredundant representation and each Q_i is primary. If $N = Q_1 \cap \cdots \cap Q_k$ is a minimal representation, then the associated primes of the factors are uniquely determined.

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Thm. Suppose R is Noetherian, M is f.g., and $N \lneq M$. If $\mathfrak{p}$ is minimal in $\mathrm{Ass}(M/N)$, then the $\mathfrak{p}$-primary component of N in any minimal primary decomposition is N^{ec} along $M \to M_{\mathfrak{p}}$.

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Write $IJ = Q_1 \cap \cdots \cap Q_r \cap Q_{r+1} \cap \cdots \cap Q_k$, where Q_i is $\mathfrak{p}_i$-primary, $\mathfrak{p}_i \supseteq I$ for $1 \leq i \leq r$ and $\mathfrak{p}_j \not\supseteq I$ for $r < j \leq k$.

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(c) R is an integral domain.

Sketch of proof.

Write $IJ = Q_1 \cap \cdots \cap Q_r \cap Q_{r+1} \cap \cdots \cap Q_k$, where Q_i is $\mathfrak{p}_i$-primary, $\mathfrak{p}_i \supseteq I$ for $1 \leq i \leq r$ and $\mathfrak{p}_j \not\supseteq I$ for $r < j \leq k$. If $y_j \in I \setminus \mathfrak{p}_j$ for $r < j \leq k$, then $x \in J$ implies $xy_j \in IJ$ implies $xy_j \in Q_j$ implies $x \in Q_j$. Thus $J \subseteq Q_{r+1} \cap \cdots \cap Q_k$. Also, since $I \subseteq \mathfrak{p}_i$ for $1 \leq i \leq r$, we have $I^m \subseteq Q_1 \cap \cdots \cap Q_r$.

The Krull Intersection Theorem

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