Commutative Algebra (MATH 6150)

The structure of $\langle \mathbf{Ideal}(R); \cap, +, \cdot\rangle$

Some definitions.

(1) A lattice $\langle L; \cap, +\rangle$ is *distributive* if it satisfies $x \cap (y+z) = x \cap y + x \cap z$ for all $x, y, z \in L$. (Equivalently, if it satisfies $x + (y \cap z) = (x+y) \cap (x+z)$ for all $x, y, z \in L$.)
(2) A lattice $\langle L; \cap, +\rangle$ is *modular* if it satisfies $x \cap (y+z) = x \cap y + x \cap z$ whenever $x \geq y$.
(3) An element $r \in R$ is *nilpotent* if $r^n = 0$ for some n.
(4) An ideal $I \lhd R$ is *nilpotent* if $I^n = 0$ for some n.
(5) An ideal $I \lhd R$ is *nil* if each of its elements is nilpotent.
(6) An element $e \in R$ is *idempotent* if $e^2 = e$.
(7) An ideal $I \lhd R$ is *idempotent* if $I^2 = I$ for some n.
(8) I is *abelian over* J if $I^2 \subseteq J$.
(9) The *annihilator of* I *modulo* J is $(J : I) = \{r \in \mid rI \subseteq J\}$.
(10) The *Wedderburn radical* is the largest nilpotent ideal of R (if it exists).
(11) The *nilradical* is the ideal of nilpotent elements of R.
(12) The *Jacobson radical* is the intersection of maximal ideals of R.

Some facts.

(1) (Dedekind) $\langle \mathrm{Ideal}(R); \cap, +\rangle$ is a modular lattice.
(2) (Samuel) $\langle \mathrm{Ideal}(R); \cap, +\rangle$ is a distributive lattice iff the generalized Chinese Remainder Theorem holds in R.
(3) If $I = \langle G\rangle$ and $J = \langle H\rangle$, then $IJ = \langle\{gh \mid g \in G, h \in H\}\rangle$.
(4) Ideal product is commutative, associative, has unit $= R$ and has zero $= (0)$.
(5) $IJ \subseteq I \cap J$.
(6) Ideal product distributes over arbitrary sum. ($I(\sum J_i) = \sum IJ_i$)
(7) (Ideal product in a quotient) $I/K \cdot J/K = IJ/K = (IJ + K)/K$.
(8) If $I + J = I + K = J + K$ and $I \cap J = I \cap K = J \cap K$, then $I + J$ is abelian over $I \cap J$.
(9) Any nilpotent ideal is nil. Any finitely generated nil ideal is nilpotent.
(10) R has a largest nil ideal (called the *nilradical* of R, written $\mathrm{nil}(R)$). It consists of the nilpotent elements of R.
(11) Any ideal generated by an idempotent ($I = (e), e^2 = e$) is itself idempotent ($I^2 = I$). Any finitely generated idempotent ideal is generated by an idempotent.
(12) $IJ \subseteq K$ iff $I \subseteq (K : J)$. Hence $(K : J)$ is the largest ideal that multiplies J into K.
(13) $(\cap K_i : J) = \cap(K_i : J)$ and $(K : \sum J_i) = \cap(K : J_i)$.

(14) $I + J \subseteq (IJ : I \cap J)$. Hence if $I + J = R$, then $IJ = I \cap J$.

(15) If Ideal(R) has finite height, then the following ideals are equal:
 (a) the Wedderburn radical of R
 (b) the nilradical of R.
 (c) the Jacobson radical of R.

Every covering $I \prec J$ below this radical is abelian and every covering above this radical is nonabelian. The quotient modulo the radical is a product of fields.