

COMMUTATIVE ALGEBRA

HOMEWORK ASSIGNMENT VIII

Read pages 93-99.

PROBLEMS

All rings are commutative.

1. (Hower, Martinez) Valuation subrings of \mathbb{Q} : Find (with proof) all valuation rings of the field \mathbb{Q} . Show that they are all DVR's, and give an explicit description of the valuations $v: \mathbb{Q}^* \rightarrow \mathbb{Z}$ that define these rings.

2. (Li, Praterelli) Value groups are determined by valuation rings: Suppose that $v_1: K^* \rightarrow G_1$ and $v_2: K^* \rightarrow G_2$ are two valuations on the field K whose valuation rings are equal. Assume that v_1 and v_2 are surjective. Show that there is an order-preserving isomorphism $\varphi: G_1 \rightarrow G_2$ such that $v_2 = \varphi \circ v_1$.

3. (Lizzi, Moore) Any totally ordered abelian group is a value group: Let $\langle G; +, -, 0, \leq \rangle$ be a totally ordered abelian group. Let X be the monoid of symbols $x^g, g \in G$, with multiplication and unit defined by $x^g x^h = x^{g+h}$ and $x^0 = 1$. Let \mathbb{F} be a field, let $\mathbb{F}[X]$ be the monoid ring over \mathbb{F} , and let $\mathbb{F}(X)$ be the field of fractions of $\mathbb{F}[X]$.

(a) Show that every nonzero element of $\mathbb{F}(X)$ has the form

$$\alpha = x^a \frac{\sum e_i x^{g_i}}{\sum f_j x^{h_j}}$$

where

- (i) $\sum e_i x^{g_i}, \sum f_j x^{h_j} \in \mathbb{F}[X]$ and $g_i, h_j \geq 0$ in G .
 - (ii) $\sum e_i x^{g_i}, \sum f_j x^{h_j}$ have nonzero constant terms. (If $g_i = 0$, then $e_i \neq 0$ and if $h_i = 0$, then $f_i \neq 0$.)
- (b) Show that the function $v: \mathbb{F}(X)^* \rightarrow G: \alpha \mapsto a$ is a (surjective) valuation with value group G .

4. (Batchelder, Jones) Characterizing maximal chains of primes: Let $\langle X; \leq \rangle$ be a linearly ordered set. Let X be a basis for a free abelian group $F = \bigoplus_{x \in X} \mathbb{Z}x$. Order F lexicographically. Let R be a valuation ring whose value group is isomorphic to $\langle F; +, -, 0, \leq \rangle$. (The previous problem guarantees that such an R exists.)

- (a) Show that the chain of prime ideals of R is isomorphic to the lattice of order ideals of $\langle X; \leq \rangle$.
- (b) Show conversely that any maximal chain of primes in a commutative ring is isomorphic to the lattice of order ideals of some linearly ordered set.

(Hint for (b): Let S be a commutative ring and let P be a maximal chain of primes in S . For any $s \in S$ let \mathfrak{p}_s be the least $\mathfrak{p} \in P$ containing s . Let $X = \{\mathfrak{p}_s \mid s \in S\}$ ordered by inclusion. Now describe an order-preserving bijection between $\langle P; \subseteq \rangle$ and the lattice of ideals of $\langle X; \subseteq \rangle$.)

5. (Gern, Stanton) Suppose that v is a valuation on K . Show that if $r_1 + r_2 + \cdots + r_n = 0$ in R and $n > 1$, then $v(r_i) = v(r_j)$ for some $i \neq j$. (To make this valid even when some $r_i = 0$ assume that $v(0) = \infty$.)

6. (Keller, Selker) Suppose that R_1, \dots, R_n are DVR's of the field K . Show that $R = \bigcap_{i=1}^n R_i$ is a Noetherian semilocal domain. (Hint: First show that in any infinite sequence r_1, r_2, r_3, \dots of elements of R there is some k such that $r_k \in (r_1) \cup \cdots \cup (r_{k-1})$.)

7. (Chriestensen, Strider) Show that every ideal in a Dedekind domain can be generated by " $1\frac{1}{2}$ " elements. This means that if $(0) \neq I \triangleleft R$, then for any $a \in I \setminus \{0\}$ there exists $b \in I$ such that $I = (a, b)$. (Hint: First show that if R is a Dedekind domain and $(0) \neq J \triangleleft R$, then R/J is a principal ideal ring.)

8. (Moorhead, Tuley) Show that a Dedekind domain is a UFD iff it is a PID.