Assignment VIII, Problem 8

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Show that a Dedekind domain is a ufd if and only if it is a pid.

Proof. It is true for all rings that all PIDs are UFDs. Suppose that A is a Dedekind domain and also a UFD.

To prove the other direction, suppose $\mathfrak{m}$ is a maximal ideal of A; we wish to demonstrate that $\mathfrak{m}$ is principal. let $\{x_i\}$ be a set of generators of $\mathfrak{m}$ of smallest size. Consider x_1; as an element of a UFD we can decompose into irreducibles, $x_1 = p_{1,1} \cdots p_{1,n}$. In UFDs, maximal ideals are prime and so at least one $p_{1,i} \in \mathfrak{m}$; replace x_1 with $p_{1,i}$ in the set of generators. Repeat for each generator, leaving us with a list of irreducible generaors. Furthermore, since A is a UFD, irreducible elements are prime; prime elements generate prime ideals, so if x_1 and x_2 were generators replace by $p_{1,i}$ and $p_{2,j}$ respectively, then $(p_{1,i})$ and $(p_{2,j})$ are prime ideals contained in $\mathfrak{m}$. However, prime ideals are maximal in Dedekind domains, so no prime ideal is properly contained in $\mathfrak{m}$, so $(p_{1,i}) = (p_{2,j}) = \mathfrak{m}$. But we assumed that the original $\{x_i\}$ is minimal, so it must consist of only one generator, making $\mathfrak{m}$ principal.

Now suppose that $\mathfrak{s}$ is an ideal of A, with unique prime factorization $P_1 ... P_i$. Prime ideals in Dedekind domains are maximal. Therefore $\mathfrak{s}$ is a product of maximal ideals, which have been shown to be principal. The product of principal ideals is principal, therefore every ideal of A is principal.

$\square$