Bryce Chriestensen
Zachary Strider McGregor-Dorsey
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Homework VIII, Problem 7. *Let R be a Dedekind domain. If $(0) \neq I \lhd R$, then for any nonzero $a \in I$, there exists an element $b \in I$ such that $I = (a, b)$.*

Proof. We first prove a couple lemmas.

Lemma (1)**.** *If R is a Dedekind domain and $P \lhd R$ is a prime ideal, then R/P^r is a principal ideal ring for every natural number r.*

Proof. Recall from Theorem 9.3 [AM69] that the local ring R_P is a discrete valuation ring. Thus, by Proposition 9.2 [AM69], we know that P is a principal ideal of R_P and all other nonzero ideals of R_P are powers of P. Thus, by the lattice isomorphism theorem, we know that $R_P/P^r R_P$ is a principal ideal ring.

Recall that localization is an exact functor, so application of that functor the short exact sequence

$$0 \to P^r \to R \to R/P^r \to 0$$

or R-modules gives us that $R_P/P^r R_P \cong (R/P^r)_P$ as R_P-modules and, as easily seen, as R_P-algebras. But R/P^r is already a local ring with maximal ideal P, so all elements outside of P are already units, so $(R/P^r)_P = R/P^r$.

Thus $R/P^r \cong R_P/P^r R_P$, proving the lemma. □

Lemma (2)**.** *If R is a Dedekind domain and $(0) \neq I \lhd R$, then R/I is a principal ideal ring.*

Proof. Because R is a Dedekind domain, we can write $I = P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n}$ where each P_i is a prime ideal of R, each P_i is distinct and each r_i is a natural number.

Now, consider $P_i^{r_i} + P_j^{r_j}$ where $i \neq j$. This is the smallest ideal containing both $P_i^{r_i}$ and $P_j^{r_j}$. Thus, every prime ideal in its ideal factorization must appear in $P_i^{r_i}$ and $P_j^{r_j}$. Since P_i and P_j are distinct, uniqueness of prime ideal factorization implies that its prime ideal factorization is 1, i.e. $P_i^{r_i}$ and $P_j^{r_j}$ are coprime.

Thus we can apply the Chinese Remainder Theorem to get that

$$R/I \cong R/P_1^{r_1} \times R/P_2^{r_2} \times \cdots \times R/P_n^{r_n}.$$

Now note the intersection of any ideal of the ring on the right with one of its factors is an ideal of that factor. Because each factor is a principal ideal ring by Lemma (1), these intersections are principal ideals. We immediately find that the original ideal is then a product of principal ideals, which is, of course, a principal ideal. Thus we have that R/I is a principal ideal ring. □

We are prepared to finish the proof. Let I be any nonzero ideal of R and a a nonzero element of I. Then, by Lemma (2), we have that $R/(a)$ is a principal ideal ring. Because $(a) \subset I$, it makes sense to consider the ideal $I/(a)$ of $R/(a)$. This is a principal ideal generated by, say, $b + (a)$. So, if $c \in I$, then $c + (a) = rb + (a)$ for some $r \in R$, i.e. $c - rb \in (a)$. Thus there is some $s \in R$ such that $c - rb = sa$, when $c = sa + rb \in (a, b)$. Clearly $(a, b) \subset I$, so $(a, b) = I$. The proof is complete. □

References

[AM69] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Westview Press, 1969.