Problem 6 (Keller, Selker)**.** *Suppose that $R_1, \dots, R_\ell$ are DVRs of the field K. Then $R = \bigcap_{i=1}^{\ell} R_i$ is a Noetherian semilocal domain.*

Proof. Clearly R is an integral domain, as R is a subring of a field. Let $v_i \colon K \to \mathbb{Z}$ be the valuation associated with the ring R_i. We have that

$$R_i = \{x \in K \colon v_i(x) \geq 0\}.$$

Thus for every $r \in R$ and every i we have $v_i(r) \in \mathbb{N} = \{0, 1, 2, \dots\}$. Now suppose that $v_i(r) \leq v_i(s)$ for all i. Then there are elements $u_i \in R_i$ such that $ru_i = s$. Manipulating these equations in K gives $u_i = sr^{-1} \in K$ for all i. Then $sr^{-1} = u_i \in R_i$ for all i. Thus $sr^{-1} \in R$, so in fact $s \in (r) \lhd R$. Let $\langle r_n \rangle$ be a sequence of elements from R. We claim:

$$\exists k \, \forall i > k \, \exists j \leq k \, r_i \in (r_j). \tag{1}$$

To prove (1) we will construct the index k. For each i let $m_i = \min\{v_i(r_n) : n \in \omega\}$, and let n_i be minimal such that $v_i(r_{n_i}) = m_i$. Define $F = \{r_{n_i} : i \in \{1, 2, \dots, \ell\}\}$. For each $i = 0, 1, \dots, \ell$, let $k_i = \max\{v_i(r) : r \in F\}$. Define a map $v \colon R \to \mathbb{N}^\ell$ by

$$v(r) = (v_1(r), v_2(r), \dots, v_\ell(r)).$$

There is a partial order on $\mathbb{N}^\ell$ defined by $(x_1, x_2, \dots x_\ell) \leq (y_1, y_2, \dots, y_\ell)$ whenever $x_i \leq y_i$ for all $i \in \{1, 2 \dots, \ell\}$. We have seen above that $v(x) \leq v(y)$ implies that $y \in (x) \lhd R$. Now consider the subset $T \subseteq \mathbb{N}^\ell$ defined by

$$T = \{m_1, \dots, k_1 - 1\} \times \{m_2, \dots, k_2 - 1\} \times \cdots \times \{m_\ell, \dots, k_\ell - 1\}$$

Clearly the set T is finite. Now for each $t \in T$, define $S_t = \{r_i \in \langle r_n \rangle : v(r_i) = t\}$. For each $t \in T$, if S_t is nonempty then choose one element from that set. Let S be the collection of these choices. Then the number of elements in S is at most the number of elements of T and is therefore finite. Let N be the largest subscript occurring on any element of S. Then define

$$k = \max\{N, n_1, n_2, \dots, n_\ell\}.$$

To prove (1), suppose $i > k$. Then either $v(r_i) \geq v(r_j)$ for some $r_j \in F$ or else $v(r_i) = v(r_j)$ for some $r_j \in S$. In either case, $j \leq k$ and we have $v(r_j) \leq v(r_i)$ so $r_i \in (r_j) \lhd R$. This establishes the claim (1).

Now suppose that some ideal $I = (r_0, r_1, \dots) \lhd R$ has infinitely many generators $\langle r_n \rangle$. By (1) there is a k such that $I = (r_0, \dots, r_k)$, so R is Noetherian.

Now suppose that R contains infinitely many distinct maximal ideals,

$$M_1, M_2, M_3, \dots.$$

Note that for all n $M_n \not\subseteq \bigcup_{i<n} M_i$, as, by the prime avoidance lemma, this would imply that $M_n \subseteq M_i$ for some i. We define a sequence $\langle r_n \rangle$ inductively. Choose $r_1 \in M_1$. If r_n has been chosen, take $r_{n+1} \in M_{n+1} \setminus \bigcup_{i=1}^{n} M_i$. Note that whenever $i < j$ we have $r_j \notin (r_i) \subseteq M_i$, contradicting claim (1). Then R contains only finitely many distinct maximal ideals, and is therefore semilocal. This completes the proof. $\square$