Problem 2. (Gern, Stanton). *Suppose that v is a valuation on K, and $R \subset K$ a subring. Show that if $r_1 + r_2 + \cdots + r_n = 0$ in R and $n > 1$, then $v(r_i) = v(r_j)$ for some $i \neq j$. (To make this valid even when some $r_i = 0$ assume that $v(0) = \infty$).*

Solution. Suppose that K is a field and that $v : K^\times \to (\Gamma, \leq)$ is a valuation on K. I will show that $r_1 + r_2 + \cdots + r_n = 0$ in R for $n > 1$ implies that $v(r_i) = v(r_j)$ for some $i \neq j$ by induction on n.

Claim. $v(-1) = 0$

Proof. Suppose towards contradiction that $v(-1) \neq 0$. Then $v(-1) > 0$ or $v(-1) < 0$. If $v(-1) > 0$ then since v is a homomorphism we have

$$0 < v(-1) < v(-1) + v(-1) = v(-1 \cdot -1) = v(1) = 0,$$

a contradiction. We get a similar contradiction if $v(-1) < 0$, therefore $v(-1) = 0$. $\square$

Now consider the case where $n = 2$. Then we have $r_1 + r_2 = 0$, so $r_2 = -r_1$. Then by the claim

$$v(r_2) = v(-r_1) = v(-1 \cdot r_1) = v(-1) + v(r_1) = v(r_1).$$

Next assume that the theorem holds for all $n < k$ and suppose that we have $r_1 + r_2 + \cdots + r_k = 0$. Then $(r_1 + r_2) + r_3 + \cdots + r_k = 0$, so by the inductive hypothesis either $v(r_i) = v(r_j)$ for some $i \neq j$ with $i, j \geq 3$ as desired, or $v(r_1 + r_2) = v(r_i)$ for some $i \geq 3$. If $v(r_1) = v(r_2)$ then we are done. If $v(r_1) \neq v(r_2)$, then without loss of generality suppose that $v(r_1) > v(r_2)$. Then since v is a valuation, $v(r_1 + r_2) \geq \min\{v(r_1), v(r_2)\} = v(r_2)$. Suppose that $v(r_1 + r_2) > v(r_2)$. Then

$$v(r_2) = v((r_1 + r_2) - r_1) \geq \min\{v(r_1 + r_2), v(-r_1)\} = \min\{v(r_1 + r_2), v(r_1)\} > v(r_2),$$

a contradiction. Then $v(r_1 + r_2) = v(r_2)$, so $v(r_i) = v(r_2)$. Hence, by induction, $r_1 + r_2 + \cdots + r_n = 0$ in R and $n > 1$ implies that $v(r_i) = v(r_j)$ for some $i \neq j$.

$\square$