Commutative Algebra Homework 8

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4) Characterizing maximal chains of primes: Let $\langle X; \leq \rangle$ be a linearly ordered set. Let X be a basis for a free abelian group $F = \bigoplus_{x \in X} \mathbb{Z}x$. Order F lexicographically. Let R be a valuation ring whose value group is isomorphic to $\langle F; +, -, 0, \leq \rangle$ (the previous problem guarantees that R exists).

a) Show that the chain of prime ideals of R is isomorphic to the lattice of order ideals of $\langle X; \leq \rangle$.

b) Show conversely that any maximal chain of primes in a commutative ring is isomorphic to the lattice of order ideals of some linearly ordered set.

Solution

a) Recall that there is a correspondence between prime ideals of R and convex subgroups of F given by the map φ defined by

$$\varphi(\mathfrak{p}) = \{g \in F \colon \forall x \in \mathfrak{p}, |g| < v(x)\}$$

where v is the valuation for R. This correspondence is a order-reversing bijection, as clearly if $\mathfrak{p} \subset \mathfrak{q}$, then

$$\{g \in F \colon \forall x \in \mathfrak{p}, |g| < v(x)\} \supset \{g \in F \colon \forall x \in \mathfrak{q}, |g| < v(x)\}.$$

For each $g \in (\oplus_{x \in x} \mathbb{Z}x)^*$, define $LSupp(g)$ to be the least element of the support of g. We note that for $g \in (\oplus_{x \in x} \mathbb{Z}x)^*$, the convex subgroup generated by g contains $1LSupp(g)$ as either $-|g| \leq 1LSupp(g) \leq |g|$ or $-|g| \leq -1LSupp(g) \leq |g|$. Suppose $g, h \in (\oplus_{x \in x} \mathbb{Z}x)^*$. If $LSupp(g) < LSupp(h)$, then we have $-1 \cdot LSupp(g) \leq h \leq 1 \cdot LSupp(g)$, so h lies in the convex subgroup generated by g. If $LSupp(g) > LSupp(h)$, then for all positive integers n, we have $-|h| < -n \cdot LSupp(g)$ and $|h| \geq n \cdot LSupp(g)$, so h is not in the convex subgroup generated by g. If $LSupp(g) = LSupp(h)$, then setting $n = |h_{LSupp(h)}|$, we have $-(n+1)LSupp(g) \leq h \leq (n+1)LSupp(g)$. Hence, we have shown that h is in the convex subgroup generated by g iff $LSupp(g) \leq LSupp(h)$. In particular, we have that the convex subgroup generated by g is equal to the convex subgroup generated by $1 \cdot LSupp(g)$.

We'll show that the convex subgroups of $F = \bigoplus_{x \in X} \mathbb{Z}x$ are of the form $\bigoplus_{x \in Y} \mathbb{Z}x$ where Y is an order filter (i.e. if $y \in Y$, then $x \in Y$ for all $x \in X$ with $x > y$). Suppose Y is an order filter on X. Let $g \in F \setminus \bigoplus_{x \in Y} \mathbb{Z}x$. Then $LSupp(g) \notin Y$, so $LSupp(g) < y$ for all $y \in Y$. It follows that for all $h \in \bigoplus_{x \in Y} \mathbb{Z}x$, $h < |g|$. Therefore, $\bigoplus_{x \in Y} \mathbb{Z}x$ is a convex subgroup of F. Conversely, let G is a convex subgroup of F.

Then, by the previous paragraph, we have that G is the convex subgroup generated by $\{1 \cdot LSupp(g) | g \in G^*\}$. Also by the previous paragraph, we have $\{LSupp(g) | g \in G^*\}$ is an order filter in X, and hence $G = \bigoplus_{x \in \{LSupp(g) | g \in G^*\}} \mathbb{Z}x$

If $Y_1 \subset Y_2$ are two order filters of X, then $\bigoplus_{x \in Y_1} \mathbb{Z}x \subset \bigoplus_{x \in Y_2} \mathbb{Z}x$. Hence, we have an order preserving bijection between the chain of convex subgroups of F and the order filters of X. Composing this bijection with φ, we get an order-reversing bijection from the chain of prime ideals of R to the lattice of order filters of $\langle X; \leq \rangle$. Finally, we note that the complement map that takes a set $Y \subset X$ to $X \setminus Y$ induces an order-reversing bijection from lattice of order filters of $\langle X; \leq \rangle$ to the lattice of order ideals of $\langle X; \leq \rangle$, and so it follows that there exists an order-preserving bijection from the chain of prime ideals of R to the lattice of order filters of $\langle X; \leq \rangle$.

b) Let S be a commutative ring and let P be a maximal chain of primes in S. For any $s \in \bigcup P$ let $\mathfrak{p}_s$ be the intersection of all primes in P containing s; that is, $\mathfrak{p}_s = \bigcap \{\mathfrak{p} \in P | s \in \mathfrak{p}\}$. We'll show that $\mathfrak{p}_s$ is the least $\mathfrak{p} \in P$ containing s. For this, it suffices to show that $\mathfrak{p}_s$ is prime, and then show that $\mathfrak{p}_s \in P$.

Suppose that $\mathfrak{p}_s$ is not prime. Then there exists $x, y \in S \setminus \mathfrak{p}_s$ with $xy \in \mathfrak{p}_s$. Since $x, y \notin \mathfrak{p}_s = \bigcap \{\mathfrak{p} \in P | s \in \mathfrak{p}\}$, there exists some $\mathfrak{a}, b \in P$ such that $s \in \mathfrak{a}$, $s \in \mathfrak{b}$ but $x \notin \mathfrak{a}$ and $y \notin \mathfrak{b}$. Since P is a chain, we either have $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{b} \subset \mathfrak{a}$. Without loss of generality, suppose $\mathfrak{a} \subset \mathfrak{b}$. Since we have $s \in \mathfrak{a} \in P$, we have $\mathfrak{p}_s \subset \mathfrak{a}$. Hence, since $xy \in \mathfrak{p}_s$, $xy \in \mathfrak{a}$. Since $\mathfrak{a}$ is prime and $x \notin \mathfrak{a}$, we must have $y \in \mathfrak{a} \subset \mathfrak{b}$. However, this contradicts our choice that $y \notin \mathfrak{b}$. Therefore, are assumption that $\mathfrak{p}_s$ is not prime is false, so $\mathfrak{p}_s$ is prime.

Next, we'll show that $\mathfrak{p}_s \in P$. Let $\mathfrak{q} \in P$. If $s \notin \mathfrak{q}$, then, since P is a chain, we have that $\mathfrak{q}$ lies below all primes in P that contain s, and hence $\mathfrak{q} \subset \bigcap \{\mathfrak{p} \in P | s \in \mathfrak{p}\} = \mathfrak{p}_s$. Conversely, if $s \in \mathfrak{q}$, then we have $\mathfrak{p}_s = \bigcap \{\mathfrak{p} \in P | s \in \mathfrak{p}\} \subset \mathfrak{q}$. Hence we have that all primes in P either lie completely in $\mathfrak{p}_s$ or completely contain $\mathfrak{p}_s$. Since $\mathfrak{p}_s$ is prime and P is a maximal chain of primes, we must have $\mathfrak{p}_s \in P$.

Let $X = \{\mathfrak{p}_s | s \in \bigcup P\}$ ordered by inclusion. Note that for all primes $\mathfrak{p}$, the set $\{\mathfrak{p}_s | s \in \mathfrak{p}\}$ is an order ideal of $\langle X; \leq \rangle$. Define the map φ from $\langle P; \subseteq \rangle$ to the lattice of ideals of $\langle X; \subseteq \rangle$ by

$$\varphi(\mathfrak{p}) = \{\mathfrak{p}_s | s \in \mathfrak{p}\}.$$

This map is clearly order preserving. Further, define an inclusion-preserving map ψ from the lattice of ideals of $\langle X; \subseteq \rangle$ to P by $\psi(O) = \bigcup O$. To show that ψ is well-defined (i.e. that the image of ψ is indeed in P), Let $I = \{\mathfrak{p}_t | t \in T\}$ be an ideal in $\langle X; \subseteq \rangle$. Since $\mathfrak{p}_t \in P$ for all $t \in T$ and since P is a chain, we have $\psi(I) = \bigcup_{t \in T} \mathfrak{p}_t$ is an ideal. Let $x, y \in S$ such that $xy \in \bigcup_{t \in T} \mathfrak{p}_t$. Then $xy \in \mathfrak{p}_t$ for some $t \in T$. Since $\mathfrak{p}_t$ is prime, either $x \in \mathfrak{p}_t \subseteq \bigcup_{t \in T} \mathfrak{p}_t$ or $y \in \mathfrak{p}_t \subseteq \bigcup_{t \in T} \mathfrak{p}_t$, so we have that $\bigcup_{t \in T} \mathfrak{p}_t$ is prime. Since P is a maximal chain of primes in S, we have that $\bigcup_{t \in T} \mathfrak{p}_t \in P$.

For all primes $\mathfrak{p} \subset S$, we have

$$\psi \circ \varphi(\mathfrak{p}) = \psi\left(\{\mathfrak{p}_s | s \in \mathfrak{p}\}\right) = \bigcup_{s \in \mathfrak{p}} \mathfrak{p}_s = \mathfrak{p}.$$

Let $I = \{\mathfrak{p}_t | t \in T\}$ be an ideal of $\langle X; \subseteq \rangle$. We note that if $s \in \bigcup_{t \in T} \mathfrak{p}_t$, then $s \in \mathfrak{p}_t$ for some t, which implies that $\mathfrak{p}_s \subset \mathfrak{p}_t \in I$. Since I is an order ideal, we have $\mathfrak{p}_s \in I = \{\mathfrak{p}_t | t \in T\}$, and therefore $\mathfrak{p}_s = \mathfrak{p}_{t_s}$ for some $t_s \in T$. Hence,

$$\varphi \circ \psi(I) = \varphi\left(\bigcup_{t \in T} \mathfrak{p}_t\right) = \left\{\mathfrak{p}_s | s \in \bigcup_{t \in T} \mathfrak{p}_t\right\} = \{\mathfrak{p}_t | t \in T\} = I.$$

Therefore, ψ is the inverse of φ, and so φ is an isomorphism.