COMMUTATIVE ALGEBRA: HOMEWORK 8

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3) Any totally ordered abelian group is a value group: let $\langle G; +, -, 0; \leq \rangle$ be a totally ordered abelian group. Let X be the monoid of symbols x^g, $g \in G$, with multiplication and unit defined by $x^g x^h = x^{g+h}$ and $x^0 = 1$. Let $\mathbb{F}$ be a field, $\mathbb{F}[X]$ the monoid ring over $\mathbb{F}$, and $\mathbb{F}(X)$ the field of fractions of $\mathbb{F}[X]$.

(a) Show that every nonzero element of $\mathbb{F}(X)$ has the form

$$\alpha = x^a \frac{\sum c_i x^{g_i}}{\sum f_j x^{h_j}},$$

where $\sum c_i x^{g_i}$, $\sum f_j x^{h_j} \in \mathbb{F}[X]$ have nonzero constant terms and $g_i, h_j \geq 0$ in G.

(b) Show that the function $\nu : \mathbb{F}(X)^\times \to G : \alpha \mapsto a$ is a (surjective) valuation with value group G.

Solution

(a) *Proof.* Every nonzero element of $\mathbb{F}(X)$ can be written in the form $\alpha = (\sum c_i x^{g_i'})/(\sum f_j x^{h_j'})$ with $c_i, f_j \in \mathbb{F}$. G is totally ordered, so let $a' = \min g_i'$ and $b' = \min h_j'$. Let $g_i = g_i' - a'$, $h_j = h_j' - b'$, and $a = a' - b'$. Then

$$\alpha = \frac{\sum c_i x^{g_i'}}{\sum f_j x^{h_j'}} = \frac{x^{a'} \sum c_i x^{g_i' - a'}}{x^{b'} \sum f_j x^{h_j' - b'}} = x^a \frac{\sum c_i x^{g_i}}{\sum f_j x^{h_j}}.$$

Since $a' \leq g_i'$ and $b' \leq h_j'$, we have $0 \leq g_i$ and $0 \leq h_j$. Since a' and b' were the minima of the g_i' and h_j', respectively, there is some l and some k such that $g_l' = a'$ and $h_k' = b'$. Thus $x^{g_l} = x^{h_k} = x^0 = 1$, so the constant terms for the numerator and denominator are c_l and f_k, respectively. Since the c_i and f_j were nonzero, the claim holds. $\square$

(b) *Proof.* We begin by proving that ν is well defined. Elements of $\mathbb{F}[X]$ can be written as $\sum_{g \in G} c_g x^g$, where only finitely many c_g are nonzero. In the case where all "exponents" are positive, we can write instead $\sum_{g \geq 0} c_g x^g$. Let $\alpha, \beta \in \mathbb{F}(X)^\times$ with decomposition as in (a) given by

$$\alpha = x^a \frac{\sum_{g \geq 0} c_g x^g}{\sum_{g \geq 0} d_g x^g} \quad \text{and} \quad \beta = x^b \frac{\sum_{g \geq 0} e_g x^g}{\sum_{g \geq 0} f_g x^g}.$$

Suppose that $\alpha = \beta$. Reducing the fractions and collecting like terms, we have

$$x^{a-b} \sum_{g,h \geq 0} c_g f_h x^{g+h} = \sum_{g,h \geq 0} d_g e_h x^{g+h}$$

Since the constant terms of the numerator and denominator of α and β are nonzero, the constant terms in the left and right sums are nonzero. Thus $c_0 f_0 x^{a-b}$ is nonzero and appears on the left. Since $\mathbb{F}[X]$ is an $\mathbb{F}$-vector space with basis $\{x^g \mid g \in G\}$, there must be positive $k, l \in G$ such that $c_0 f_0 x^{b-a} = d_k e_l x^{k+l}$. In particular,

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$a-b=k+l \geq 0$, so $a \geq b$. A similar argument shows that $b-a \geq 0$ as well. Hence $a-b=0$, so $\nu(\alpha)=a=b=\nu(\beta)$. Therefore ν is well-defined.

We will now show that ν is a valuation on $\mathbb{F}(X)^{\times}$. Let $\alpha, \beta \in \mathbb{F}(X)^{\times}$ have decompositions

$$\alpha = x^a \frac{\sum_{g \geq 0} c_g x^g}{\sum_{g \geq 0} d_g x^g} \quad \text{and} \quad \beta = x^b \frac{\sum_{g \geq 0} e_g x^g}{\sum_{g \geq 0} f_g x^g}.$$

Then

$$\alpha\beta = x^{a+b} \frac{\sum_{g \geq 0}(c_g x^g) \sum_{g \geq 0}(e_g x^g)}{\sum_{g \geq 0}(d_g x^g) \sum_{g \geq 0}(f_g x^g)} = x^{a+b} \frac{\sum_{g,h \geq 0} c_g e_h x^{g+h}}{\sum_{g,h \geq 0} d_g f_h x^{g+h}}.$$

Since $g+h \geq 0$ and the constant terms, $c_0 e_0$ and $d_0 f_0$, are nonzero, this is the decomposition of $\alpha\beta$. Hence

$$\nu(\alpha\beta) = a+b = \nu(\alpha)+\nu(\beta).$$

It remains to show that $\nu(\alpha+\beta) \geq \min(\nu(\alpha), \nu(\beta))$. Without loss of generality, assume that $a \leq b$. Then

$$\begin{aligned}
\alpha+\beta &= x^a \frac{\sum_{g \geq 0} c_g x^g}{\sum_{g \geq 0} d_g x^g} + x^b \frac{\sum_{g \geq 0} e_g x^g}{\sum_{g \geq 0} f_g x^g} = x^a \left(\frac{\sum_{g \geq 0} c_g x^g}{\sum_{g \geq 0} d_g x^g} + x^{b-a} \frac{\sum_{g \geq 0} e_g x^g}{\sum_{g \geq 0} f_g x^g} \right) \\
&= x^a \frac{\sum_{g \geq 0}(c_g x^g) \sum_{g \geq 0}(f_g x^g) + x^{b-a} \sum_{g \geq 0}(d_g x^g) \sum_{g \geq 0}(e_g x^g)}{\sum_{g \geq 0}(d_g x^g) \sum_{g \geq 0}(f_g x^g)} \\
&= x^a \frac{\sum_{g,h \geq 0} c_g f_h x^{g+h} + x^{b-a} \sum_{g,h \geq 0} d_g e_h x^{g+h}}{\sum_{g,h \geq 0} d_g f_h x^{g+h}} \\
&= x^a \frac{\sum_{g,h \geq 0} c_g f_h x^{g+h} + d_g e_h x^{g+h+b-a}}{\sum_{g,h \geq 0} d_g f_h x^{g+h}}
\end{aligned}$$

The numerator and denominator have positive "exponents" and the constant term of the denominator, $d_0 f_0$, is nonzero. If $a \neq b$, then the constant term of the numerator is $c_0 f_0$, which is nonzero, so $\nu(\alpha+\beta) = a \geq \min(\nu(\alpha), \nu(\beta))$. If $a = b$ then the constant term of the numerator is $c_0 f_0 + d_0 e_0$. If $c_0 f_0 + d_0 e_0 \neq 0$ then we have $\nu(\alpha+\beta) = a \geq \min(\nu(\alpha), \nu(\beta))$. In the case where $c_0 f_0 + d_0 e_0 = 0$, we can factor an $x^{a'}$ from the numerator (as in (a)) such that the factored numerator has nonzero constant term and all positive exponents. Since all of exponents in the unfactored numerator were positive, $a' \geq 0$. Therefore $\nu(\alpha+\beta) = a + a' \geq a = \min(\nu(\alpha), \nu(\beta))$. In all cases, $\nu(\alpha+\beta) \geq \min(\nu(\alpha), \nu(\beta))$, so ν is a valuation. Let $b \in G$. Then $x^b \in F(X)$ and $\nu(x^b) = b$. Hence ν is surjective. It follows that G is a value group of $\mathbb{F}(X)$ with associated valuation ν. $\square$