Commutative Algebra Homework 8

Li, Praterelli

2) Suppose that $v_1 : K^* \to G_1$ and $v_2 : K^* \to G_2$ are two valuations on the field K whose valuation rings are equal. Assume that v_1 and v_2 are surjective. There is an order-preserving isomorphism $\varphi : G_1 \to G_2$ such that $v_2 = \varphi \circ v_1$.

proof: By definition, v_i is a homomorphism from the multiplicative group $K^* = K \setminus \{0\}$ onto $G_i = v_i(K^*) \overset{\varphi_i}{\simeq} K^*/\ \ker(v_i)$. Notice that for each x in $\ker(v_i)$ we have $v_i(x^{-1}) = -v_i(x) = 0$ so that $\ker(v_i)$ consists of units of R_{v_i}. But for each unit x of R_{v_i} we have $-v_i(x) = v_i(x^{-1}) \geq 0$ so that $0 \geq v_i(x)$ which is impossible unless $v_i(x) = 0$. Hence $\ker(v_i)$ is exactly the group of units of R_{v_i}. Because $R_{v_1} = R_{v_2}$, we have $\ker(v_1) = \ker(v_2)$. We thus get the isomorphism $\varphi = \varphi_2^{-1} \circ \varphi_1$.

We now show that φ is order preserving. Suppose a and b are in G_1 such that $a \leq b$. Because v_1 is onto there exist x and y in K^* such that $v_1(x) = a$ and $v_1(y) = b$. Hence $v_1(x) \leq v_1(y)$ and $0 \leq v_1(y) - v_1(x) = v_1(yx^{-1})$. It follows that yx^{-1} is in R_{v_1}. Since $R_{v_1} = R_{v_2}$, yx^{-1} is also in R_{v_2}, so that $0 \leq v_2(yx^{-1}) = v_2(y) - v_2(x)$ and $v_2(x) \leq v_2(y)$. But $v_2 = \varphi \circ v_1$, so that $\varphi(v_1(x)) \leq \varphi(v_1(y))$ and $\varphi(a) \leq \varphi(b)$.