Problem 1 Valuation subrings of $\mathbb{Q}$: Find (with proof) all valuation rings of the field $\mathbb{Q}$. Show that they are all DVR's, and give an explicit description of the valuations $v : \mathbb{Q} \to \mathbb{Z}$ that define these rings.

Proof Let $(G, \leq)$ be an ordered group and $v : \mathbb{Q} \to G$ be a valuation.

$v(1) = 0$ and $0 = v(1) = v(a\frac{1}{a}) = v(a) + v(\frac{1}{a})$ so $v(\frac{1}{a}) = -v(a)$ thus $v(-1) = 0$ and so v(-a)=v(a).

if $\frac{p}{q} \in \mathbb{Q}$ then $v(\frac{p}{q}) = v(p) - v(q)$ so it suffices to determine where v maps the integers.

if every prime is mapped to 0, then v is trivial. Suppose then that $v(p) = x \in G$ for some fixed prime $p > 0$.

For any integer q, $v(q) = v(1 + 1 + \ldots + 1) \geq \min(v(1), v(1), \ldots, v(1)) = 0$.

Let $q > 0$ be an arbitrary prime which is not equal to p. We may use the Euclidean algorithm to find integers r, s such that $1 = pr + qs$. Note that p does not divide s, otherwise, p would divide 1. Applying v to both sides, since $v(t) \geq 0$ for all integers t we therefore have

$$0 = v(1) = v(pr + qs) \geq \min(v(p) + v(r), v(q) + v(s)) \geq 0.$$

$v(p)$ is nonzero, so it must be that $v(q) = 0 = v(s)$. We thus have that $v(p^n) = nx$ so the image of v is isomorphic to $\mathbb{Z}$ via $x \mapsto 1$, furthermore, this is an order preserving isomorphism. Hence any nontrivial valuation of $\mathbb{Q}$ must be a discrete valuation determined by a prime p as follows: $v_p(p^n \frac{a}{b}) = n$ where a and b are relatively prime to p.

Since the valuation rings are the positive cone of a valuation we thus have that every valuation ring of $\mathbb{Q}$ is of the form $\{\frac{a}{b} \in \mathbb{Q} : p \nmid b\}$ for a fixed prime p.

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