COMMUTATIVE ALGEBRA: HOMEWORK 7 PROBLEM 7

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Problem 7. Show that, if $\mathbb{F}$ is a field, then every subring, R, of $\mathbb{F}[x]$ containing $\mathbb{F}$ is Noetherian.

Proof. First note that if $R = \mathbb{F}$ then we are done since fields are Noetherian. Otherwise we will use the following result:

Proposition: Let $A \subseteq B \subseteq C$ be rings. Suppose that A is Noetherian, that C is finitely generated as an A-algebra and C is finitely generated as a B-module. Then B is finitely generated as an A-algebra.

We have the situation $\mathbb{F} \subseteq R \subseteq \mathbb{F}[x]$ with $\mathbb{F}$ Noetherian since it is a field, and $\mathbb{F}[x]$ is finitely generated as an $\mathbb{F}$-algebra. We will show that $\mathbb{F}[x]$ is finitely generated as an R-module. Once we know that R is a finitely generated $\mathbb{F}$-algebra then it is Noetherian by a corollary to the Hilbert Basis Theorem.

We need only to show that $\mathbb{F}[x]$ is finitely generated as an R-module. Let $p \in R$ be any monic non constant polynomial of degree d. Let $a_i \in \mathbb{F}$ for $i = 1, 2, \cdots, d-1$ such that $p = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$. Let $\mathfrak{B} = \{1, x, \cdots, x^{d-1}, x^d\}$ and B be the R-span of $\mathfrak{B}$ in $\mathbb{F}[x]$. Note that $x^{d+1} = xp - a_{d-1}x^d - \cdots - a_1 x^2 - a_0 x \in B$ because $\mathbb{F} \subseteq R$. Now let $N \geq d$ and assume that $x^n \in B$ for all $n < N$. Then since $\mathbb{F}[x]$ is a Euclidean domain we have that there are unique $q, r \in \mathbb{F}[x]$ with $r = 0$ or $\deg(r) < \deg(p) = d$ such that $x^N = pq + r$. Since $\deg(p) \geq 0$ we have that $\deg(q) < N$ so $q \in B$ and since $p \in R$ we have $qp \in B$. Also since $\deg(r) \leq d$ we have $r \in B$. Hence $x^N \in B$. Therefore by induction we have that $x^n \in B$ for all $n \in \mathbb{N}$. Thus $\mathbb{F}[x] \in B$, making $\mathbb{F}[x]$ a finitely generated R-module. By what we said above we have now shown that R is Noetherian.