Commutative Algebra Homework 7, Problem 6
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Part(a) Determine which rings have the property that the canonical map $R \to R_S$ is surjective for all multiplicatively closed sets S.

Claim 1. *A ring R has the property that the function $R \to R_S$ mapping r to $r/1$ is surjective for all multiplicatively closed sets S if and only if for all $f \in R$, the chain $(f) \supseteq (f^2) \supseteq (f^3) \supseteq \cdots$ terminates.*

Proof. Suppose that for all multiplicatively closed sets S, the map $R \to R_S$ is surjective. Let $f \in R$, and consider the multiplicatively closed set $S = \{1, f, f^2 \cdots\}$. Since the map $R \to R_S$ is surjective and $1/f \in R_S$, there exists $r \in R$ such that $1/f = r/1$. Hence, there exists $m \geq 0$ such that $f^m(1-rf) = 0$, or equivalently, $f^m = f^{m+1}r$. Thus, f^{m+1} divides f^m, so $(f^{m+1}) \supseteq (f^m)$. On the other hand, f^m clearly divides f^{m+1}, so $(f^{m+1}) \subseteq (f^m)$. Thus, $(f^m) = (f^{m+1})$. Furthermore, $(f^{m+2}) = (f)(f^{m+1}) = (f)(f^m) = (f^{m+1})$, so an easy inductive proof shows that

$$(f^m) = (f^{m+1}) = (f^{m+2}) = \cdots .$$

Thus, the chain

$$(f) \supseteq (f^2) \supseteq (f^3) \supseteq \cdots$$

terminates.

To prove the reverse implication, suppose that for all $f \in R$, the chain $(f) \supseteq (f^2) \supseteq (f^3) \supseteq \cdots$ terminates. Let S be a multiplicatively closed set, and let $f \in S$. Then since the chain $(f) \supseteq (f^2) \supseteq \cdots$ terminates, there exists $m \geq 0$ such that $(f^m) = (f^{m+1})$. Hence, f^{m+1} divides f^m, so there exists $r \in R$ such that $f^m = f^{m+1}r = f^m(fr)$. Thus, $f^m(1 - fr) = 0$. Now, S is multiplicatively closed and $f \in S$, so $f^m \in S$. Therefore, $r/1 = 1/f$ in R_S. Thus, if $a/f \in R_S$, then $(ar)/1 = (a/1)(r/1) = (a/1)(1/f) = a/f$, so we conclude that the map $R \to R_S$ is surjective.

□

Part (b) Conclude that every Artinian ring has this property.

If R is Artinian, then *any* descending chain of ideals terminates. Thus, for all $f \in R$, the chain $(f) \supseteq (f^2) \supseteq (f^3) \supseteq \cdots$ terminates. By part (a), R has the desired property.

Part (c) Give an example of a non-Artinian ring that has this property.

Consider the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ of countably many copies of $\mathbb{Z}_2$. Then we can consider each element of R as a function $f : \mathbb{N} \to \mathbb{Z}_2$ under pointwise multiplication and addition. For any $x \in \mathbb{N}$ and $n > 0$, $f(x)^n = f(x)$, since $f(x) = 0$ or $f(x) = 1$. Thus, for all $f \in R$ and $n > 0$, $f^n = f$. Hence, the chain $(f) \supset (f^2) \supset \cdots$ terminates immediately. By part (a), the map $R \to R_S$ is surjective for all multiplicatively closed sets R. Note, however, that R is not Artinian. Let $I_1 = \{f \in R : f(1) = 0\}$, $I_2 = \{f \in R : f(1) = f(2) = 0\}$, and so on. Then the chain $I_1 \supset I_2 \supset I_3 \supset \cdots$ never terminates, so R is not Artinian.