Problem 5 (Hower, Selker)**.** *Let A be an integral domain and K its field of fractions. Call $x \in K$* almost integral *over A if $\exists a \in A \setminus \{0\}$ such that $ax^n \in A$ for all n.*
(a) If $x \in K$ is integral over A then x is almost integral over A.
(b) If A is Noetherian, then any $x \in K$ almost integral over A is integral over A.

Proof. (a) Suppose that $x \in K$ is integral over A. Say x is a root of a degree n monic polynomial with coefficients in A, then it suffices to show that for some $a \in A \setminus \{0\}$, $ax^k \in A$ for all $k < n$. As K is the field of fractions of A we have $x = rs^{-1}$ for some $r \in A$ and $s \in A \setminus \{0\}$. Then $s^n \in A \setminus \{0\}$ and if $k < n$, $s^n x^k = s^{n-k} r \in A$.

(b) Suppose that A is Noetherian and that $x \in K$ is almost integral over A. Thus we may fix $a \in A$ such that $ax^n \in A$ for all n. Now $a^{-1}A$ is finitely generated as an A module so $a^{-1}A$ is Noetherian. We claim that $A[x] \leq a^{-1}A$. Now $A[x]$ is generated as an A-module by monic monomials, x^n, and by almost-integrality, $ax^n \in A$, so $a^{-1}ax^n = x^n \in a^{-1}A$ for all n. Thus $A[x]$ is an A-submodule of $a^{-1}A$, so it must be finitely generated, say by $\{f_0, \ldots, f_n\}$. Now taking $m > \max\{\deg f_i : i = 0, \ldots, n\}$ we have that $x^m = c_0 f_0 + \ldots + c_n f_n$, hence $x^m - c_0 f_0 - \ldots - c_n f_n = 0$. By choice of m, the above polynomial is monic, so x is integral over A. $\square$