Commutative Algebra Homework 7

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4) The *Integral Closure* $\overline{I}$ of I is the set of elements of A integral over I.

a) Show that $I \subseteq \overline{I} \subseteq \sqrt{I}$.

b) Give examples to show that $I \neq \overline{I}$ and $\overline{I} \neq \sqrt{I}$ are possible.

Solution

a) Let I be an ideal of a ring A. An element $r \in A$ is integral over I if there exists an integer n and elements $a_i \in I^i$, $i = 1, \cdots, n$ such that $r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$. Let $r \in I$, $n = 1$ and $a_1 = -r$. Then $r + (-r) \cdot 1 = 0$. Therefore $r \in \overline{I}$ and $I \subseteq \overline{I}$.
Let $r \in \overline{I}$. From the definition of r being integral over I, $r^n \in (a_1, \cdots, a_n) \subseteq I$. Therefore $\overline{I} \subseteq \sqrt{I}$.

b) i) *Example where* $I \neq \overline{I}$: Let $A = \mathbb{Z}[x, y]$ and let $I = (x^2, y^2)$. Let $n = 2$, $a_1 = 0 \in (x^2, y^2)$ and $a_2 = -x^2 y^2 \in (x^2, y^2)^2$. Then $(xy)^2 + a_1(xy) + a_2 = 0$ which implies that $xy \in \overline{(x^2, y^2)}$. However, $xy \notin (x^2, y^2)$. Therefore $I \neq \overline{I}$.

ii) *Example where* $\overline{I} \neq \sqrt{I}$: Let $A = \mathbb{Z}[\sqrt{2}]$ and let $I = (\{2\sqrt{2}\})$. $\sqrt{2} \in \sqrt{I}$ because $\sqrt{2} \in A$ and $(\sqrt{2})^6 = 8 \in I$. We claim that $\sqrt{2} \notin \overline{I}$ (and therefore $\overline{I} \neq \sqrt{I}$.) To see that $\sqrt{2} \notin \overline{I}$, suppose otherwise. That is, suppose $\sqrt{2}$ satisfies a polynomial $x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ for some integer n and elements $a_i \in I^i$, $i = 1, ..., n$. We get

$$(\sqrt{2})^n = -(a_1(\sqrt{2})^{n-1} + \cdots + a_{n-1}\sqrt{2} + a_n).$$

Observe that $I^i = (\{(2\sqrt{2})^i\})$. We get $a_i = m_i (2\sqrt{2})^i$, where m_i is a ring element. Hence

$$a_i(\sqrt{2})^{n-i} = m_i(2\sqrt{2})^i(\sqrt{2})^{n-i} = m_i 2^i (\sqrt{2})^n$$

so that

$$(\sqrt{2})^n = -(\sqrt{2})^n \textstyle\sum_{i=1}^n m_i 2^i = -(\sqrt{2})^{n+2} \sum_{i=1}^n m_i 2^{i-1}.$$

Calculating in $\mathbb{Q}(\sqrt{2})$ we get $\frac{1}{2} = -\sum_{i=1}^n m_i 2^{i-1}$. This is a contradiction because $-\sum_{i=1}^n m_i 2^{i-1} \in \mathbb{Z}[\sqrt{2}]$ but $\frac{1}{2}$ is not. Hence $\sqrt{2} \notin \overline{I}$ (and therefore $\overline{I} \neq \sqrt{I}$.)