COMMUTATIVE ALGEBRA: HOMEWORK 7

MATTHEW MOORE, JOHN TULEY, AND YINGWEI LI

2) Suppose that $A \leq B$ is an integral extension and that B is finitely generated as an A-algebra. Show that for every prime $\mathfrak{p} \in \operatorname{Spec}(A)$ there are only finitely many primes of B lying over $\mathfrak{p}$.

Solution

Lemma 1. *Let $A \leq B$ be an integral extension and suppose that B is finitely generated as an A-algebra. If $\mathfrak{m} \lhd A$ is maximal then there are only finitely many primes of B lying over $\mathfrak{m}$.*

Proof. B is finitely generated, so write $B = A[x_1, \ldots, x_n]$. Then $A/\mathfrak{m}$ is a field and $C = (A/\mathfrak{m})[x_1, \ldots, x_n]$ is a integral extension. Since C is finitely generated as an $A/\mathfrak{m}$-algebra and is integral over $A/\mathfrak{m}$, it is finitely generated as an $A/\mathfrak{m}$-vector space. Hence C satisfies DCC on its subspace lattice. Since every ideal of C is a subspace, C is an Artinian ring. Artinian rings have only finitely many prime ideals, so in particular there are only finitely many primes lying over 0. The primes of C lying over 0 in $A/\mathfrak{m}$ are in bijective correspondence with the primes of B lying over $\mathfrak{m}$ in A. $\square$

Proof of Problem. B is finitely generated as an A-algebra, so say $B = A[x_1, \ldots, x_n]$. Let $\mathfrak{p} \in \operatorname{Spec}(A)$. Then $A_{\mathfrak{p}} \leq B_{\mathfrak{p}} = A_{\mathfrak{p}}[x_1, \ldots, x_n]$ is an integral extension and $A_{\mathfrak{p}}$ is local with maximal ideal $\mathfrak{p}^e$. By the above lemma, there are only a finite number of primes in $B_{\mathfrak{p}}$ lying over $\mathfrak{p}^e$.

Let $\alpha : A \to A_{\mathfrak{p}} : a \mapsto a$ and $\beta : B \to B_{\mathfrak{p}} : b \mapsto b$ be the canonical homomorphisms. Under these homomorphisms, there is an inclusion-preserving bijection between the primes of $B_{\mathfrak{p}}$ and the primes of B disjoint from $A \setminus \mathfrak{p}$. The primes of B disjoint from $A \setminus \mathfrak{p}$ are those primes $\mathfrak{q} \lhd B$ such that $\mathfrak{q} \cap A \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a contracted prime, by restricting to maximal primes we have a bijection between primes of $B_{\mathfrak{p}}$ lying over $\mathfrak{p}^e$ in $A_{\mathfrak{p}}$ and primes of B lying over $\mathfrak{p}$ in A. $\square$

Date: November 19, 2009.