Math 6150, Assignment 7, Problem 1

Adam Lizzi and Michael Martinez

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#1. Let A be an integrally closed integral domain. Write K for the field of fractions of A. Show that for monic polynomials in $A[x]$, irreducibility when viewed in $A[x]$ corresponds with irreduciblity when viewed in $K[x]$.

Proof. Let f be such a monic polynomial in $A[x]$. Surely if f is irreducible over K, it is irreducible over A. ($A[x]$ is a subset of $K[x]$ – polynomials in A had their chance to divide f in this larger context, and yet none of them did.) In the other direction, let f be irreducible over A. We must show it remains irreducible when viewed over K. Assume this wasn't the case. Since K is a field, $K[x]$ is a principal ideal domain, ergo a unique factorization domain. Thus we can break f into irreducibles in $K[x]$; let g be an arbitrary one of them. If g weren't monic, but $f = gh$ for some $h \in K[x]$, the product of the leading coefficients of g and h will be the leading coefficient of f, namely 1. Using α and β for the leading coefficients of g and h, we have $\alpha = \beta^{-1}$. Thus $\alpha\beta f = f = (\beta g)(\alpha h)$, and these two factors are now monic. Continue by induction if h wasn't irreducible; we see that we can assume that all the factors of f in $K[x]$ are monic themselves.

Let L denote an algebraic closure of K. Recall that the set $\overline{A}$ of elements of L integral over A is in fact a ring. Every root of g (in L specifically) is a root of f, and every root of f in L lies in $\overline{A}$ since it satisfies a monic polynomial with coefficients in A. So the roots of g are also integral over A. Let $r_1, \cdots, r_n$ be the roots of g in L. In $L[x]$, we have $g = (x - r_1)\cdots(x - r_n)$. Multiply these binomials together to recover g. The coefficients that result are sums of products of the roots (specifically they're symmetric polynomials applied to the roots), and because $\overline{A}$ is a ring, sums of products of the r_i are in $\overline{A}$. Thus all the coefficients of g are integral over A. Since $g \in K[x]$, the coefficients of g are in K as well. As A is integrally closed, the coefficients of g must live in A. We've deduced $g \in A[x]$. This argument applies to every irreducible factor of f in $K[x]$, so if $f = gh$ in $K[x]$, then both g and h will be in $A[x]$ (h may not be irreducible but it will be the product of things in $A[x]$), and this contradicts the irreducibility of f in $A[x]$. So the notion of irreducibility over A does coincide with the notion over K. $\square$