Assignment VI, Problem 7 Stanton, Strider, Tuley

7 *Let R be an infinite Noetherian integral domain of cardinality ρ that has a maximal ideal of index κ.*

(a) *Use the Krull Intersection Theorem to prove that $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$.*

Solution. First we show that $\kappa + \aleph_0 \leq \rho$.

If κ is finite, then $\kappa + \aleph_0$ is simply $\aleph_0$, because $\aleph_0$ is the larger cardinality. Because R is infinite, its cardinality must be as large as or larger than the smallest infinite cardinality, $\aleph_0$. So $\kappa + \aleph_0 = \aleph_0 \leq \rho$ as needed.

If κ is infinite, then κ is as large as or larger than $\aleph_0$, again because it is the smallest infinite cardinality, whence $\kappa + \aleph_0 = \kappa$. Let M be the maximal ideal of R with index κ, so the ring R/M has cardinality κ. The natural map from R to R/M is a surjective map, so the cardinality of R must be as large as or larger than that of R/M, i.e. $\rho \geq \kappa = \kappa + \aleph_0$ as needed.

Second we show that $\rho \leq \kappa^{\aleph_0}$.

We claim that if R/M has cardinality $\kappa \geq \aleph_0$, then so does R/M^t for any natural number t. We will prove this by induction.

The ring R/M has cardinality κ by our assumption, so the base case is proved. Now fix t and assume that R/M^r has cardinality κ for all $r < t$. Consider the natural map from R/M^t to R/M^{t-1} which sends $r + M^t$ to $r + M^t + M^{t-1} = r + M^{t-1}$ for all r in R. The kernel of this map is clearly all $r + M^t$ where $r \in M^{t-1}$, i.e. the ideal M^{t-1}/M^t of R/M^t. Thus we can calculate the the cardinality of R/M^t to be the cardinality of the image R/M^{t-1} multiplied by the cardinality of the kernel M^{t-1}/M^t.

We know, by assumption, the cardinality of R/M^{t-1} is κ. We also know that, because R is Noetherian, M^{t-1} (and hence M^{t-1}/M^t) is finitely generated. If a is a generator of M^{t-1} and r is in M, then ra is zero in M^{t-1}/M^t, so M^{t-1}/M^t is generated by the sums of its generators with coefficients in R/M. That is, M^{t-1}/M^t is a finitely generated module of R/M. Because the cardinality of R/M is κ, the cardinality of M^{t-1}/M^t is at most $n\kappa$ where n is the (finite) number of generators of M^{t-1}/M^t. Since $\kappa \geq \aleph_0$, we know $n\kappa = \kappa$.

Thus we have that the cardinality of R/M^t is $\kappa\kappa = \kappa$ as needed.

Note that the same argument shows that if R/M has finite cardinality, then so does R/M^t. (It just might be a different finite number.)

Finally, we look at the natural map

$$\pi : R \to \prod_{t=1}^{\infty} R/M^t.$$

The kernel of this map is any of element of R that is sent to each M^t for all natural numbers t. By Krull's Intersection Theorem, we know the only such element of R is 0. So we have the π is an injective map. But then the cardinality of R must be less than or equal to that of $\prod_{t=1}^{\infty} R/M^t$, which since each term has cardinality κ, is simply $\kappa^{\aleph_0}$. (Note that this still works is κ is finite, as the cardinality will be a product of $\aleph_0$ finite natural numbers, which is $2^{\aleph_0}$. We can easily show this with the diagonal argument.) So we have the the cardinality ρ of R is less than or equal to $\kappa^{\aleph_0}$ and we are done. ∎

(b) *Show that if ρ and κ are infinite cardinals satisfying $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$ then there is a Noetherian integral domain R of cardinality ρ with a maximal ideal of index κ.* [1]

Solution. Let ρ and κ be infinite cardinals satisfying $\kappa + \aleph_0 \leq \rho \leq \kappa^{\aleph_0}$. Let $\mathbb{F}$ be a field of cardinality κ.

We claim that $\mathbb{F}[x]$ has cardinality $\kappa + \aleph_0$. For each nonnegative integer m, let S_m be the set of polynomials over $\mathbb{F}$ of degree m. For a given m, S_m has cardinality κ, since S_m can be put in one-to-one correspondence with the Cartesian product of $m + 1$ copies of $\mathbb{F}$. Thus, for each m, there exists a

[1] Hint: consider rings R such that $\mathbb{F}[x] \subseteq R \subseteq F[[x]]$ where $\mathbb{F}$ is a field of cardinality κ.

bijection $g_m : S_m \to \mathbb{F}$. Now, define a map $g : \bigcup_{m=0}^{\infty} S_m \to (\mathbb{N} \cup \{0\}) \times \mathbb{F}$ taking $x \in S_m$ to $(m, g_m(x))$. Since each g_m is bijective, g is also bijective. Now $(\mathbb{N} \cup \{0\}) \times \mathbb{F}$ has cardinality $\aleph_0 \kappa = \kappa$. Hence, $\mathbb{F}[x] = \bigcup_{m=0}^{\infty} S_m$ has cardinality κ, as well. Since κ is infinite, $\kappa + \aleph_0 = \kappa$.

Now, we show that $\mathbb{F}[[x]]$ has cardinality $\kappa^{\aleph_0}$. Each element $z(x) = \sum_{n=0}^{\infty} a_n x^n$ corresponds bijectively to a function $f_z : \mathbb{N} \cup \{0\} \to \mathbb{F}$ defined by $f_z(n) = a_n$. Thus, the cardinality of $\mathbb{F}[[x]]$ is $\left|\mathbb{F}^{\mathbb{N} \cup \{0\}}\right| = \kappa^{\aleph_0}$, by the definition of $\kappa^{\aleph_0}$.

Because $|\mathbb{F}[x]| = \kappa + \aleph_0$ and $|\mathbb{F}[[x]]| = \kappa^{\aleph_0}$, we seek a Noetherian ring R of cardinality ρ such that $\mathbb{F}[x] \subseteq R \subseteq \mathbb{F}[[x]]$. Since $\kappa + \aleph_0 \le \rho \le \kappa^{\aleph_0}$, there exists a set S_0 of cardinality ρ such that $\mathbb{F}[x] \subseteq S_0 \subseteq \mathbb{F}[[x]]$. Let S_1 be the subring of $\mathbb{F}[[x]]$ generated by the elements of S. Since there are finitely many finitary ring operations, $|S_0| = |S_1| = \rho$. Let U_1 be the set of elements u in S_1 whose inverse u^{-1} lies in $\mathbb{F}[[x]]$ (these will be all power series in S_1 with nonzero constant term). Now, let S_2 be the ring generated by the set $S_1 \cup \{u^{-1} : u \in U\}$. Again, R has cardinality ρ. Let S_3 be the ring generated by $S_2 \cup \{u^{-1} : u \in U_2\}$, where U_2 is the set of elements in S_2 that are invertible in $\mathbb{F}[[x]]$. Continue in this way, and let $S = \bigcup_{n=1}^{\infty} S_n$. Then S is a ring with cardinality ρ, since S_n has cardinality ρ for all $n \in \mathbb{N}$.

We claim that the ideal lattice of R is isomorphic to the ideal lattice for $\mathbb{F}[[x]]$. We follow the idea of the proof of problem 1, part 2, from assignment 1. Let I be a proper nonzero ideal of R. Since I is a proper ideal, every element of I is not a unit, and thus has zero constant term. For each element $p(x) = \sum_{n=0}^{\infty} a_n x^n$ of I, let d_p be the smallest n such that $a_n \neq 0$. Let N be the smallest element of the set $\{d_p : p(x) \in I\}$. We claim that $I = (x^N)$. Let $p(x) = \sum_{n=0}^{\infty} a_n x^n$ be in I. Then $a_n = 0$ for all $n < N$, and thus, $p(x)$ is divisible by x^N. Hence, $p(x) \in (x^N)$, and we conclude that $I \subseteq (x^N)$. On the other hand, by the definition of N, there exists $q(x) = \sum_{n=0}^{\infty} b_n x^n$ in I such that $a_n = 0$ for all $n < N$ and $a_N \neq 0$. Thus,

$$q(x) = x^N (a_N + a_{N+1} x + a_{N+2} x^2 + \cdots),$$

and since $a_N \neq 0$, $a_N + a_{N+1} x + a_{N+2} x^2 + \cdots$ is a unit. Hence, $x^N \in I$, since I is an ideal. Therefore, $I = (x^N)$.

It is easy to see that $(x^m) \subseteq (x^n)$ if and only if $m \ge n$. Thus, the ideal lattice of R is isomorphic to the lattice of the Noetherian ring $\mathbb{F}[[x]]$, so R is Noetherian. Now, (x) is a maximal ideal in R, and $R/(x) \cong \mathbb{F}$. Hence (x) has index κ in R. Thus, R is a Noetherian integral domain of cardinality ρ containing a maximal ideal of index κ, such that $\kappa + \aleph_0 \le \rho \le \kappa^{\aleph_0}$. ∎

(c) *Give examples to show that it is not possible to bound ρ in terms of κ if we drop either the hypothesis that R is Noetherian or the hypothesis that R is an integral domain.*

Solution. Let $S = \mathbb{F}_2 \times F[[x]]$, where F is a field of cardinality η such that $\eta^{\aleph_0} > 2^{\aleph_0}$. Note that S is not an integral domain, but S is Noetherian by part (b), and because products of Noetherian rings are Noetherian. Now, the maximal ideal $\{0\} \times F[[x]]$ has index $\kappa = 2$. However, if we let ρ be the cardinality of S,

$$\rho \ge \eta^{\aleph_0} > 2^{\aleph_0} = \kappa^{\aleph_0}.$$

Thus, ρ is not bounded above by $\kappa^{\aleph_0}$.

Let $R = \mathbb{Z}_2[X]$, where X is a set of commuting indeterminates x_α such that $|X| > 2^{\aleph_0}$. R is an integral domain by an argument given in Homework I, Problem 9, Part (a). Note that R is not Noetherian, because X contains a countably infinite subset $\{x_{\alpha_n} : n \in \mathbb{N}\}$, and $(x_{\alpha_1}) \subseteq (x_{\alpha_1}, x_{\alpha_2}) \subseteq \cdots$ forms an infinite ascending chain of ideals. Define a map $\Phi : R \to \mathbb{Z}_2$ that maps a polynomial to its constant term. Then Φ is clearly a homomorphism, and $\ker \Phi$ is the set of all polynomials with zero constant term. Note that Φ is surjective, since the constant polynomials 0 and 1 lie in R, so by the first isomorphism theorem,

$$R / \ker \Phi \cong \mathbb{Z}_2.$$

Since $\mathbb{Z}_2$ is a field of order 2, $\ker \Phi$ is a maximal ideal of index $\kappa = 2$. Thus, if ρ is the cardinality of R, we conclude that the following is true:

$$\rho \ge |X| > 2^{\aleph_0} = \kappa^{\aleph_0}.$$

Thus, ρ is not bounded above by $\kappa^{\aleph_0}$. ∎