Problem 6 (Praterelli, Selker)**.** *Let M be a f.g. subdirectly irreducible (SI) module over a Noetherian ring R.*

(a) M is Artinian.

(b) M has a composition series and all composition factors are isomorphic.

Proof. (a) Because M is SI, the zero submodule is meet-irreducible. So (0) is $\mathfrak{p}$-primary for some prime $\mathfrak{p}$, and $\operatorname{Ass}(M) = \{\mathfrak{p}\}$. Let M_0 be the minimal nonzero submodule of M. Then $\emptyset \neq \operatorname{Ass}(M_0) \subseteq \operatorname{Ass}(M) = \{\mathfrak{p}\}$, so $\operatorname{Ass}(M_0) = \{\mathfrak{p}\}$. Then $R/\mathfrak{p} \hookrightarrow M_0$. By minimality, this embedding is also surjective, so $R/\mathfrak{p} \cong M_0$, in particular $R/\mathfrak{p}$ is simple, so $\mathfrak{p}$ is maximal. Now as $\mathfrak{p}$ is the only associated prime of M and M is finitely generated we have that for some n, $\mathfrak{p}^n = (0 : M)$. Thus the action on M by $R/\mathfrak{p}^n$ is faithful, and we may view M as an $R/\mathfrak{p}^n$-module. Now $\operatorname{Ideal}(R/\mathfrak{p}^n)$ has finite height, so $R/\mathfrak{p}^n$ is Artinian. Thus M is Artinian as a f.g. module over $R/\mathfrak{p}^n$.

(b) As M is Artinian and Noetherian, any strict chain of submodules must be finite. Let $0 \leq M_0 \leq \ldots \leq M_n = M$ be a maximal strictly ascending chain of submodules of M. By maximality, this chain is a composition series. By part (a) $M_0 \cong R/\mathfrak{p}$. We claim that also $M_{i+1}/M_i \cong R/\mathfrak{p}$ for all i. Because it is a composition factor, M_{i+1}/M_i is simple. Also, M_{i+1}/M_i has an associated prime, $\mathfrak{q} = \sqrt{(M_i : M_{i+1})}$. Then if $r \in \mathfrak{q}$, $\lambda_r : M_{i+1} \to M_{i+1}$ is an endomorphism which if injective would be surjective because the modules are Artinian. However, $\operatorname{im} \lambda_r \subseteq M_i < M_{i+1}$ thus $\lambda_r : M \to M$ is not injective, so $r \in \mathfrak{p}$. Thus $\mathfrak{q} \subseteq \mathfrak{p}$. Conversely, by simplicity of M_{i+1}/M_i, we have that $\mathfrak{q}$ is maximal so in fact $\mathfrak{q} = \mathfrak{p}$, whence $M_{i+1}/M_i \cong R/\mathfrak{p}$. $\square$