COMMUTATIVE ALGEBRA: HOMEWORK 6

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5) Prove in each of the following ways that a subdirectly irreducible Noetherian ring must be Artinian:

(a) Using primary decomposition: show that R has a unique associated prime, which is a nilpotent maximal ideal. Then show that a Noetherian ring with a nilpotent maximal ideal is Artinian.

(b) Using the Krull Intersection Theorem: again, first show that R has a nilpotent maximal ideal.

Solution

Lemma 1. If R is a ring with a maximal nilpotent ideal then R is Noetherian if and only if it is Artinian.

Proof. Let N be a maximal nilpotent ideal. Then there is n such that $N^n = 0$. Therefore we have descending chain $N \supseteq N^2 \supseteq N^3 \supseteq \ldots \supseteq N^n = 0$. N^m/N^{m+1} is a module over R/N for $1 \leq m < n$. Since R/N is a field, N^m/N^{m+1} is a R/N-vector space and hence ACC is equivalent to DCC in each factor. Suppose that R is Noetherian. Then N^m/N^{m+1} satisfies ACC and hence DCC. Since every factor satisfies DCC, it must be that R satisfies DCC and is thus Artinian. Suppose that R is Artinian. Then N^m/N^{m+1} satisfies DCC and hence ACC. Since each factor satisfies ACC, it must be that R satisfies ACC and is thus Noetherian. □

(a) *Proof.* If R is a field then the claim holds. Assume that R is not a field. Let A be the unique least nonzero ideal of R and let $a \in A \setminus 0$. Since A is the least ideal, we have that $A = (a)$. Considering A as an R-module, the primary ideals are those primes in R that annihilate A. Consider $\mathfrak{p} = (0 : a)$ and suppose that $x \notin (0 : a)$. Then $xa \neq 0$, so since $A = (a)$ is the least ideal, $(a) = (xa)$. Thus there is y such that $a = xya$, so $a(1 - xy) = 0$. Therefore $1 - xy \in (0 : a)$. It follows that $(0 : a) + (x)$ contains 1, and $(0 : a) = \mathfrak{p}$ is maximal and hence prime. Furthermore, $\mathrm{Ass}(A) = \{\mathfrak{p}\}$ and A is $\mathfrak{p}$-primary.

We will now show that $\mathfrak{p}$ is nilpotent. Since R is Noetherian, it is enough to show that $\mathfrak{p}$ is nil. Suppose that $b \in \mathfrak{p} = (0 : a)$ and $b^n \neq 0$ for all n. The ideals $(0 : b) \subseteq (0 : b^2) \subseteq \ldots$ are an ascending chain, so since R is Noetherian there must be some $n \in \mathbb{Z}_{>0}$ such that $(0 : b^n) = (0 : b^{n+1})$. $b^n \neq 0$, so $(a) \subseteq (b^n)$. Hence there is $r \in R$ such that $rb^n = a$. Therefore $rb^{n+1} = ab = 0$, and so $r \in (0 : b^{n+1}) = (0 : b^n)$. This implies that $a = rb^n = 0$, a contradiction. Hence b is nilpotent, so $b \in \mathrm{nil}(R)$. Thus $(0 : a) = \mathfrak{p} \subseteq \mathrm{nil}(R)$. Since $\mathfrak{p}$ is maximal and $1 \notin \mathrm{nil}(R)$, we have that $\mathfrak{p} = \mathrm{nil}(R)$ and is thus nil. R is Noetherian, so nil implies nilpotent. Therefore $\mathfrak{p}$ is maximal and nilpotent. Using the lemma, we can conclude at this point that R is Artinian. However, it remains to show that $\mathfrak{p}$ is unique.

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Consider R to be a module over itself. The uniqueness and minimality of A implies that the zero ideal is meet irreducible. This in turn implies that (0) is a primary ideal. It is a fact that if $N \subsetneqq M$ and N is $\mathfrak{p}$-primary then $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$. Replacing M by R and N by (0) yields that $\mathrm{Ass}(R)$ has one element, which must be $\mathfrak{p}$.

$\square$

(b) *Proof.* If R is a field then the claim holds. Assume that R is not a field. As above, let $A = (a)$ be the unique least nonzero idea of R. Since $(a) \neq 0$, $(0 : a) \neq R$. Hence there is maximal ideal $\mathfrak{m}$ such that $(0 : a) \subseteq \mathfrak{m}$ (R is not a field, so such a proper maximal ideal exists). By the Krull Intersection Theorem, there is $m \in \mathfrak{m}$ such that $(1 - m)\bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0$. Since $A = (a)$ is the smallest ideal, there are two possibilities for $\bigcap \mathfrak{m}^i$: either $(a) \subseteq \bigcap \mathfrak{m}^i$, or $\bigcap \mathfrak{m}^i = 0$. Suppose that $(a) \subseteq \bigcap \mathfrak{m}^i$. Then

$$0 = (1 - m)\bigcap_{i=1}^{\infty} \mathfrak{m}^i \supseteq (1 - m)(a).$$

Hence $1 - m \in (0 : a) \subseteq \mathfrak{m}$. Since $m \in \mathfrak{m}$, this implies that $1 \in \mathfrak{m}$, contradicting maximality. Therefore $(a) \not\subseteq \bigcap \mathfrak{m}^i$. It follows that $\bigcap \mathfrak{m}^i = 0$. If $(a) \subseteq \mathfrak{m}^i$ for all i, then $(a) \subseteq \bigcap \mathfrak{m}^i = 0$, a contradiction. Thus there is some n such that $(a) \not\subseteq \mathfrak{m}^n$, so $\mathfrak{m}^n = 0$. Therefore $\mathfrak{m}$ is maximal and nilpotent, so by the above lemma R is Artinian. $\square$