Math 6150, Assignment 6, Problem 4

Adam Lizzi and Michael Martinez

November 10, 2009

#4. Show:

(a) if $0 \to L \to M \to N \to 0$ is exact, then $\mathrm{Supp}(M) = \mathrm{Supp}(L) \cup \mathrm{Supp}(N)$. (The union of the supports is the support of an extension of N by L.)

(b) $\mathrm{Supp}(L) \cap \mathrm{Supp}(N) = \mathrm{Supp}(L \otimes N)$. (The intersection of the supports is the support of the tensor product.)

(c) (Unrelated.) Show that if R is Noetherian, its total ring of fractions is semilocal. (The total ring of fractions is the ring R_S where $S \subseteq R$ is the set of elements that aren't zero divisors; it's the largest set for which the canonical map $R \to R_S$ is an embedding.) (A semilocal ring has finitely many maximal ideals.)

Proof. (a) Localization is exact, so let's localize our exact sequence:

$$0 \to L_{\mathfrak{p}} \to M_{\mathfrak{p}} \to N_{\mathfrak{p}} \to 0.$$

Assume $\mathfrak{p} \in \mathrm{Supp}(L) \cup \mathrm{Supp}(N)$. Then either $M_{\mathfrak{p}}$ accepts an injection from a nonzero module or admits a surjection to a nonzero module, so $M_P \neq 0$; conclude $\mathfrak{p} \in \mathrm{Supp}(M)$. Now assume $\mathfrak{p} \in \mathrm{Supp}(M)$. Assume further that $L_{\mathfrak{p}} = 0$. Then the map $M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is a surjection with trivial kernel, ergo a bijection. Since $M_{\mathfrak{p}}$ was assumed to be nontrivial, $N_{\mathfrak{p}}$ is nontrivial as well. So at least one of $L_{\mathfrak{p}}$ or $N_{\mathfrak{p}}$ is nontrivial, so $\mathfrak{p} \in \mathrm{Supp}(L) \cup \mathrm{Supp}(N)$.

(b) Use ca5p5, which says that for modules over a local ring, the tensor product is zero if and only if one of the factors is. (Technically ca5p5 only asserts the forward direction, but obviously if one of the factors is zero the tensor product will be.) For a prime $\mathfrak{p}$, the modules $L_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ are modules over a local ring. Since $L_{\mathfrak{p}} \otimes N_{\mathfrak{p}} = (L \otimes N)_{\mathfrak{p}}$, we have what we want. Conclude $\mathfrak{p} \in \mathrm{Supp}(L \otimes N)$ if and only if $\mathfrak{p} \in \mathrm{Supp}(L)$ and $\mathfrak{p} \in \mathrm{Supp}(N)$, so that $\mathrm{Supp}(L \otimes N) = \mathrm{Supp}(L) \cap \mathrm{Supp}(N)$ as desired.

(c) This fact will follow by judiciously considering R as a module over itself. Since R is Noetherian, finitely generated modules over R will have finitely many associated primes. R is 1-generated over itself. So $\mathrm{Ass}(R)$ is a finite collection of primes of R. Further, the union of the associated primes of a module over a Noetherian ring is the set of zero divisors on that module. In the current situation the zero divisors on R are simply the set $R \backslash S$, with S the set of non-zero divisors of R. As with any localization, the primes of R_S are in one to one correspondence with the primes of R disjoint from S. Primes of R disjoint from S land in the set $R \backslash S = \bigcup \mathrm{Ass}\, R$. We'll show there are finitely many maximal ideals in $\bigcup \mathrm{Ass}\, R$, and by the correspondence this will imply that R_S has finitely many maximal ideals and thus is semilocal. This fact follows readily from the prime avoidance lemma. It states that any prime contained in $\bigcup \mathrm{Ass}\, R$ must be contained in a single prime $\mathfrak{p} \in \mathrm{Ass}\, R$. Therefore the only ideals in $\bigcup \mathrm{Ass}\, R$ that could be maximal are the associated primes themselves, and there are finitely many of those. Ergo R_S is semilocal. $\square$