COMMUTATIVE ALGEBRA HOMEWORK VI

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Problem 3 Throughout this problem R is Noetherian, M is a f.g. R-module and N is an arbitrary R-module and N is an arbitrary R-module.

(a) Assume in addition that R is local with maximal ideal $\mathfrak{m}$. Show that $\mathfrak{m} \in \operatorname{Ass}(\operatorname{Hom}_R(M, N))$ iff $M \neq 0$ and $\mathfrak{m} \in \operatorname{Ass}(N)$.

(b) Now drop the assumption that R is local. Use part (a) and localization to prove that

$$\operatorname{Ass}(\operatorname{Hom}_R(M, N)) = \operatorname{Supp}(M) \cap \operatorname{Ass}(N).$$

Solution

(a) Suppose that $\mathfrak{m} \in \operatorname{Ass}(\operatorname{Hom}_R(M, N))$. Then we have an R-module isomorphism:

$$R/\mathfrak{m} \cong \langle \varphi \rangle \leq \operatorname{Hom}_R(M, N)$$

for some homomorphism $\varphi \in \operatorname{Hom}_R(M, N)$. Since $R/\mathfrak{m}$ is a ring (indeed, a field) it contains 1. Since $\mathfrak{m} \neq R$, it follows that $1 \neq 0$ in $R/\mathfrak{m}$. Then by the above isomorphism, $\langle \varphi \rangle$ contains a nonzero element and hence φ itself must be nonzero. If $M = 0$ then there could be no such nonzero homomorphism $\varphi\colon M \to N$, so we must have $M \neq 0$. By an alternative characterization of associated primes we have that $\mathfrak{m} = (0\colon \varphi)$. Suppose $x \in \mathfrak{m}$. Since $\varphi(M)$ is nonzero, there exists $n \in \varphi(M)$ with $n \neq 0$. Choose $m \in M$ so that $\varphi(m) = n$. Then we have

$$xn = x(\varphi(m)) = (x\varphi)(m) = 0$$

since $x\varphi$ is the zero map in $\operatorname{Hom}_R(M, N)$. Then $x \in (0\colon n)$ and hence $\mathfrak{m} \subseteq (0\colon n)$. But $(0\colon n) \neq R$ since $1 \in R$ acts on N as the identity endomorphism, so that $1n = n \neq 0$. Then we must have $\mathfrak{m} = (0\colon n)$ by the maximality of $\mathfrak{m}$. Then $\mathfrak{m} \in \operatorname{Ass}(N)$.

Now begin with the supposition that $M \neq 0$ and $\mathfrak{m} \in \operatorname{Ass}(N)$. Since $M \neq 0$ there exists $m \in M$ with $m \neq 0$. Since M is finitely generated, we may choose a minimal generating set $B = \{m_0, m_1, m_2, \ldots, m_k\}$ where $m = m_0$. Since B is a generating set, a map $\varphi\colon B \to N$ will extend linearly to a homomorphism $\overline{\varphi}\colon M \to N$ provided that

$$a_0\varphi(m) + a_1\varphi(m_1) + \cdots + a_k\varphi(m_k) = 0$$

in N whenever

$$a_0 m + a_1 m_1 + \cdots + a_k m_k = 0$$

in M, where $a_i \in R$. We define a map $\varphi\colon B \to N$ as follows. First choose $n \in N$ so that $\mathfrak{m} = (0\colon n)$, possible since $\mathfrak{m} \in \operatorname{Ass}(N)$. Then $n \neq 0$. Then define $\varphi(m) = n$ and $\varphi(m_i) = 0$ for $i = 1, 2, \ldots, k$. Suppose there exist a_i such that

$$a_0 m + a_1 m_1 + \cdots + a_k m_k = 0.$$

Then we have

$$a_0\varphi(m) + a_1\varphi(m_1) + \cdots + a_k\varphi(m_k) = a_0\varphi(m) = a_0 n.$$

Then φ will extend to a homomorphism $\overline{\varphi}\colon M \to N$ if $a_0 n = 0$. Suppose a_0 is a unit. Then m can be expressed in terms of the other $k-1$ generators via

$$m = (a_0)^{-1}(-a_1 m_1 - a_2 m_2 - \cdots - a_k m_k)$$

contradicting the minimality of B. Then a_0 is not a unit and hence $(a_0) \lhd R$ is a proper ideal. Since R is a local ring, (a_0) must be contained in the unique maximal ideal $\mathfrak{m} = (0\colon n)$. Then $a_0 n = 0$ as required. Then $\overline{\varphi}$ is a nonzero element of $\operatorname{Hom}_R(M, N)$ with image $\langle n \rangle$. Then the ideal $(0\colon \varphi)$ is precisely the ideal $(0\colon n)$ and hence $\mathfrak{m} = (0\colon \varphi)$ and we have $\mathfrak{m} \in \operatorname{Ass}(\operatorname{Hom}_R(M, N))$ as required. This completes the proof. $\square$

(b) We require two preliminary results, which we will prove before the main result:

Claim 1 Let R be a Noetherian ring and M an R-module. Then $\mathfrak{p} \in \operatorname{Ass}(M)$ iff $\mathfrak{p}_\mathfrak{p} \in \operatorname{Ass}(M_\mathfrak{p})$.

Proof. Suppose $\mathfrak{p} \in \operatorname{Ass}(M)$. Then $\mathfrak{p} = (0\colon m)$ for some element $m \in M$. Let $p/s \in \mathfrak{p}_\mathfrak{p}$ where $p \in \mathfrak{p}$ and $s \in S = R - \mathfrak{p}$. Then we have

$$\frac{p}{s} \cdot \frac{m}{1} = \frac{pm}{s} = \frac{0}{s} = 0 \in M_\mathfrak{p}.$$

Then $p/s \in (0\colon m/1)$. Then $\mathfrak{p}_\mathfrak{p} \subseteq (0\colon m/1) \lhd R_\mathfrak{p}$. Suppose $m/1 = 0/1$. Then there exists $t \in S$ such that $tm = 0$. But this is a contradiction, since $t \notin \mathfrak{p} = (0\colon m)$. Then $m/1$ is not annihilated by $1 = 1/1 \in R_\mathfrak{p}$ and therefore $(0\colon m/1)$ is a proper ideal. But $\mathfrak{p}_\mathfrak{p}$ is maximal in the local ring $R_\mathfrak{p}$ so we must have $\mathfrak{p}_\mathfrak{p} = (0\colon m/1)$. Then $\mathfrak{p}_\mathfrak{p} \in \operatorname{Ass}(M_\mathfrak{p})$.

Now suppose that $\mathfrak{p} \notin \operatorname{Ass}(M)$ and that $\mathfrak{p}_\mathfrak{p} \in \operatorname{Ass}(M_\mathfrak{p})$. Then $\mathfrak{p}_\mathfrak{p} = (0\colon m/s)$ where $m \in M$ and $s \in S$. Since $\mathfrak{p} \notin \operatorname{Ass}(M)$ we have that $\mathfrak{p} \neq (0\colon m)$. We consider several cases:

Case 1: $(0\colon m) \not\subset \mathfrak{p}$
In this case, choose $s' \in S$ such that $s'm = 0$. Then we compute

$$\frac{s'}{1} \cdot \frac{m}{s} = \frac{s'm}{s} = \frac{0}{s}$$

and hence $s'/1 \in \mathfrak{p}_\mathfrak{p} = (0\colon m/s)$. Then $s'/1 = p/s''$ for some $p \in \mathfrak{p}$ and $s'' \in S$. Then there exists $t \in S$ such that

$$t(s's'' - p) = 0$$

and hence

$$s's'' = tp \in \mathfrak{p},$$

a contradiction since $s's'' \in S$, a multiplicatively closed set.

Case 2: $(0\colon m) \subsetneq \mathfrak{p}$
Since R is noetherian, $\mathfrak{p}$ is finitely generated. Let $\{p_1, p_2, \dots, p_k\}$ be a generating set for $\mathfrak{p}$. Since $(0\colon m)$ is properly contained in $\mathfrak{p}$, some of the generators may lie in $(0\colon m)$, but not all of them. Relabel the generators so that the first ℓ generators lie in m. Then $0 \leq \ell < k$. For each generator p_i with $i > \ell$ we have that

$$\frac{p_i}{1} \cdot \frac{m}{s} = 0$$

since $\mathfrak{p}_\mathfrak{p} = (0\colon m/s)$, so there exists $s_i \in S$ such that $s_i p_i m = 0$. Then $s_i \in (0\colon p_i m)$. Suppose that

$$A := \bigcap_{i > \ell} (0\colon p_i m) \subseteq \mathfrak{p}.$$

Then in particular we have

$$\prod_{i>\ell} s_i \in \prod_{i>\ell} (0\colon p_i m) \subseteq \bigcap_{i>\ell} (0\colon p_i m) \subseteq \mathfrak{p}$$

which is a contradiction, since S is multiplicatively closed.

It follows that there exists $s \in A$ such $s \notin \mathfrak{p}$. Then sp_i annihilates m for all $i > \ell$ and p_i annihilates m for all $i \leq \ell$. It follows that $sp \in (0\colon m)$ for all $p \in \mathfrak{p}$. Equivalently, $\mathfrak{p} \subsetneq (0\colon sm)$ (we cannot have equality here since we have assumed $\mathfrak{p} \notin \mathrm{Ass}(M)$. Then choose $x \in (0\colon sm) - \mathfrak{p}$. Then $(xs)m = 0$, hence $xs \in (0\colon m) \subset \mathfrak{p}$, contradicting the primality of $\mathfrak{p}$.

In both cases we reach a contradiction. Then $\mathfrak{p} \notin \mathrm{Ass}(M)$ implies $\mathfrak{p}_\mathfrak{p} \notin \mathrm{Ass}(M_\mathfrak{p})$, thus establishing our claim. $\square$

Claim 2 If M, N are isomorphic R-modules then $\mathrm{Ass}(M) = \mathrm{Ass}(N)$.

Proof. Let $\varphi\colon M \to N$ be an R-module isomorphism and suppose that $\mathfrak{p} \lhd R$ is an associated prime of M. Then $\mathfrak{p} = (0\colon m)$ for some $m \neq 0$ in M. Then $pm = 0$ for all $p \in \mathfrak{p}$. Then we have

$$p\varphi(m) = \varphi(pm) = \varphi(0) = 0$$

so $\mathfrak{p} \subseteq (0\colon \varphi(m))$. On the other hand, if $s \notin \mathfrak{p}$ then $sm \neq 0$ and hence

$$s\varphi(m) = \varphi(sm) \neq 0$$

since φ is injective. Then $s \notin (0\colon \varphi(m))$. Then we have $\mathfrak{p} = (0\colon \varphi(m))$ and hence $\mathfrak{p} \in \mathrm{Ass}(N)$.

Now suppose $\mathfrak{p} \in \mathrm{Ass}(N)$. Then $\mathfrak{p} = (0\colon n)$ for some $n \neq 0$ in N. Since φ is surjective, there exists $m \neq 0$ in M such that $\varphi(m) = n$. Then $\mathfrak{p} = (0\colon \varphi(m))$. Let $p \in \mathfrak{p}$. Then we have

$$\varphi(pm) = p\varphi(m) = 0$$

so $pm = 0$ since φ is injective. Then $\mathfrak{p} \subseteq (0\colon m)$. On the other hand, if $s \notin \mathfrak{p}$ then

$$\varphi(sm) = s\varphi(m) \neq 0$$

so $sm \neq 0$ since φ is injective. Then $s \notin (0\colon m)$. Thus $\mathfrak{p} = (0\colon m)$ and hence $\mathfrak{p} \in \mathrm{Ass}(M)$. This establishes our claim. $\square$

We now prove the main result. Suppose $\mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}_R(M, N))$. By claim 1 above, this occurs if and only if $\mathfrak{p}_\mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}_R(M, N)_\mathfrak{p})$. We have an R-module isomorphism

$$\mathrm{Hom}_R(M, N)_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$$

since R is Noetherian and M is finitely generated (see Lemma 11.32, Rotman's *Advanced Modern Algebra*). Then by claim 2, our hypothesis is equivalent to $\mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}))$. Since $\mathfrak{p}_\mathfrak{p}$ is the unique maximal ideal of the local ring $R_\mathfrak{p}$, we may use part (a) to conclude that our hypothesis is equivalent to the statement $M_\mathfrak{p} \neq 0$ and $\mathfrak{p}_\mathfrak{p} \in \mathrm{Ass}(N_\mathfrak{p})$. But $M_p \neq 0$ if and only if $\mathfrak{p} \in \mathrm{Supp}(M)$, by definition, and by claim 1 we have that $\mathfrak{p}_\mathfrak{p} \in \mathrm{Ass}(N_\mathfrak{p})$ if and only if $\mathfrak{p} \in \mathrm{Ass}(N)$. Then our hypothesis is equivalent to the statement that $\mathfrak{p} \in \mathrm{Supp}(M)$ and $\mathfrak{p} \in \mathrm{Ass}(N)$, that is, $\mathfrak{p} \in \mathrm{Supp}(M) \cap \mathrm{Ass}(N)$. This completes the proof. $\square$