Problem 2. (Gern, Hower). *(a) Assume that R is Noetherian, that M is a finitely generated R-module and that $L, N \leq M$ are submodules. Show that $L \subseteq N$ if and only if $L_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Ass(M/N)}$.*

(b) Show that any subset $U \subseteq Spec(R)$ can be $Ass(M)$ for some R-module M. Show that any finite subset $U_0 \subseteq Spec(R)$ can be the set of associated primes of some finitely generated module.

Solution: (a) For the forward implication suppose that $L \subseteq N$ and let $\mathfrak{p} \in \mathrm{Ass(M/N)}$. Then the inclusion $\phi : L \xrightarrow{\subseteq} N$ is injective. Now consider $\phi_{\mathfrak{p}} : L_{\mathfrak{p}} \to N_{\mathfrak{p}}$. Let $\frac{l}{u} \in \ker(\phi_{\mathfrak{p}})$. Then $\frac{l}{u} = \frac{0}{1}$ in $N_{\mathfrak{p}}$, so there exists some $v \in R \setminus \mathfrak{p}$ such that $v(l \cdot 1 - 0 \cdot u) = vl = 0$ in N. Then since ϕ is injective $vl = 0$ in L, so $\frac{l}{u} = \frac{0}{1}$ in $L_{\mathfrak{p}}$, so $\ker(\phi_{\mathfrak{p}}) = \{0\}$. Then $\phi_{\mathfrak{p}}$ is injective, so $L_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$.

For the reverse implication suppose that $L \not\subseteq N$. Then $N \subsetneq L+N$. Since R is Noetherian we see that $(L+N)/N$ must contain some associated prime $\mathfrak{p}$, so there exists some $x \in (L+N)\setminus N$ such that $\mathfrak{p} = (N : x)$. We see that also, $\mathfrak{p} \in \mathrm{Ass(M/N)}$. Then $L_{\mathfrak{p}}+N_{\mathfrak{p}} = (L+N)_{\mathfrak{p}}$ by corollary 3.4, so $\frac{x}{1} \in L_{\mathfrak{p}} + N_{\mathfrak{p}}$, and $\frac{x}{1} \neq \frac{0}{1}$ since there is no $u \in R \setminus \mathfrak{p}$ such that $xu = 0$. Now suppose that $\frac{x}{1} \in N_{\mathfrak{p}}$. Then $\frac{x}{1} = \frac{n}{u}$ for some $u \in R \setminus \mathfrak{p}$. Then there exists some $t \in R \setminus \mathfrak{p}$ such that $t(ux - n) = 0$, so $tux = tn$ in M. But we see that $tn \in N$, and $tu \in R \setminus \mathfrak{p}$, so $tux \notin N$, a contradiction. Therefore, $x \notin N_{\mathfrak{p}}$, so $N_{\mathfrak{p}} \subsetneq L_{\mathfrak{p}} + N_{\mathfrak{p}}$, and thus $L_{\mathfrak{p}} \not\subseteq N_{\mathfrak{p}}$.

(b) Let $U = \{\mathfrak{p}_i\}_{i \in I} \subseteq Spec(R)$, and set $M = \bigoplus_{i \in I} R/\mathfrak{p}_i$. (Note that M is finitely generated if U is a finite set.) Since $R/\mathfrak{p}_i \hookrightarrow M$ for all $i \in I$, $U \subseteq Ass(M)$.

Suppose that $\mathfrak{q} = (0 : s) \in Ass(M)$. We can write $s = (r_i + \mathfrak{p}_i)_{i \in I}$, where $r_i \in \mathfrak{p}_i$ except for finitely many indices $i_1, ..., i_n$.

Let $t \in \mathfrak{q}$. Then for all $i \in I$, $tr_i \in \mathfrak{p}_i$. In particular, since each $\mathfrak{p}_{i_j}$ is prime, and each $r_{i_j} \notin \mathfrak{p}_{i_j}$, we have $t \in \mathfrak{p}_{i_j}$ for $1 \leq j \leq n$. Thus $\mathfrak{q} \subseteq \bigcap_{j=1}^{n} \mathfrak{p}_{i_j}$.

On the other hand, if $t \in \bigcap_{j=1}^{n} \mathfrak{p}_{i_j}$, then $t \in \mathfrak{q}$, since $tr_{i_j} \in \mathfrak{p}_{i_j}$ for $1 \leq j \leq n$. Thus $\mathfrak{q} = \bigcap_{j=1}^{n} \mathfrak{p}_{i_j}$.

Now $\mathfrak{q} = \bigcap_{j=1}^{n} \mathfrak{p}_{i_j} \supseteq \prod_{j=1}^{n} \mathfrak{p}_{i_j}$. I have assumed that $\mathfrak{q}$ is a prime ideal, so we must have that some $\mathfrak{p}_{i_k}$ is contained in $\mathfrak{q}$. On the other hand, $\mathfrak{q}$ is the intersection of the $\mathfrak{p}_{i_j}$'s, so $\mathfrak{q} \subseteq \mathfrak{p}_{i_k}$. Thus $\mathfrak{q} = \mathfrak{p}_{i_k} \in U$, and so $Ass(M) = U$.

□