COMMUTATIVE ALGEBRA HOMEWORK VI

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Problem 1

Let $R = \mathbb{F}[x, y]$ where $\mathbb{F}$ is a field. Let $I = (x^2, xy)$, $P = (x)$, and $Q_a = (x^2, y - ax)$, $a \in \mathbb{F}$. Prove that $I = P \cap Q_a$ is a minimal primary decomposition of I for any $a \in \mathbb{F}$. Prove that $Q_a \neq Q_b$ if $a \neq b$. Find the associated primes of the primary ideals in the decompositions $I = P \cap Q_a$ and identify which are minimal.

Solution: Let $a \in \mathbb{F}$. Since $x^2, xy \in P$, $I \subseteq P$. Also, $x^2 \in Q_a$ and $xy = x(y - ax) + ax^2 \in Q_a$, so $I \subseteq Q_a$. Hence, $I \subseteq P \cap Q_a$. Suppose $f(x, y) \in P \cap Q_a$. Since $f(x, y) \in Q_a$, there exist $g(x, y), h(x, y) \in R$ with $f(x, y) = x^2 g(x, y) + (y - ax)h(x, y) = x[xg(x, y) - ah(x, y)] + yh(x, y)$. Since $f(x, y) \in (x)$, we must have that $x|h(x, y)$, say $h(x, y) = x\hat{h}(x, y)$. Then we have $f(x, y) = x[xg(x, y) - ax\hat{h}(x, y)] + xy\hat{h}(x, y) = x^2[g(x, y) - a\hat{h}(x, y)] + xy\hat{h}(x, y) \in I$. Hence, $P \cap Q_a \subseteq I$, so $P \cap Q_a = I$.

P is prime, hence also primary. To show that Q_a is primary, we'll use the following Lemma:

Lemma: If J is an ideal of R such that $\sqrt{J}$ is maximal, then J is primary.

Proof. Let $a, b \in R$ with $ab \in J$. We need to show that either $a \in J$ or $b \in \sqrt{J}$. If $b \in \sqrt{J}$, we're done. So suppose $b \notin \sqrt{J}$. Since $\sqrt{J}$ is maximal, $R/\sqrt{J}$ is a field, so b is invertible modulo $\sqrt{J}$. Let $bc = 1 - j$ where $c \in R$ and $j \in \sqrt{J}$. Choose m so that $j^m \in J$. Then $bc(1 + j + \cdots + j^{m-1}) = (1 - j)(1 + j + \cdots + j^{m-1}) = 1 - j^m$, so b is invertible modulo J. Hence $ab \in J \implies a \in J$. $\square$

We already have shown that $xy \in Q_a$. Hence, $y^2 = y(y - ax) + axy \in Q_a$, so $x, y \in \sqrt{Q_a} \implies (x, y) \subseteq \sqrt{Q_a}$. Since $R/(x, y) = \mathbb{F}$, (x, y) is a maximal ideal of R. It follows that $\sqrt{Q_a} = (x, y)$ and, by the Lemma, Q_a is primary. Therefore, $I = P \cap Q_a$ is a primary decomposition of I for any $a \in \mathbb{F}$. Since $P \neq Q_a$, $I \neq P$, and $I \neq Q_a$ we have that this is a minimal primary decomposition of I for any $a \in \mathbb{F}$.

If for any $a, b \in \mathbb{F}$ we have that $Q_a = Q_b$ then there are $u(x, y), v(x, y) \in R$ such that $y - ax = u(x, y)x^2 + v(x, y)(y - bx)$, which yields, $y[1 - v(x, y)] = x[xu(x, y) - bv(x, y) + a]$. Thus we have $x|[1 - v(x, y)]$. Let $\hat{v}(x, y) \in R$ such that $v(x, y) = 1 + x\hat{v}(x, y)$. Then we have that $(xy)\hat{v}(x, y) - x^2[u(x, y) - b\hat{v}(x, y)] = x(a - b)$. So we must have $x(a - b) \in I$. Since $x \notin I$ we must have $a - b = 0$. Thus $a = b$. Therefore if $a \neq b$ we have $Q_a \neq Q_b$.

For all $a \in \mathbb{F}$ we have $\sqrt{P} = P$ and $\sqrt{Q_a} = (x, y)$. Thus the associated primes to the decomposition $I = P \cap Q_a$ are (x) and (x, y). Since $(x) \subseteq (x, y)$ we have that (p) is a minimal associated prime, and (x, y) is an embedded prime.