Problem 8. (Chriestenson, Hower)

- Show that R is Noetherian iff Spec(R) has a cover of principal open sets $\{X_f\}$ such that each R_f is a Noetherian ring.
- Is it true that R is Noetherian iff $R_{\mathfrak{p}}$ is Noetherian for all $\mathfrak{p}$?

Solution. ($\Rightarrow$) For any $f \in R$ we have that every ideal of R_f is an extended ideal along $R \to R_f$. Hence if R is Noetherian then so is R_f. Therefore the conclusion is obvious.

($\Leftarrow$) We note that since Spec(R) is compact so we have that any cover of Spec(R) by open sets may be reduced to a finite cover. Hence we may assume there are $f_1, f_2, \ldots, f_n \in R$ such that $\{X_{f_i}\}$ is a cover of Spec(R) by principal open sets such that the R_{f_i} are Noetherian rings for all $i = 1, 2, \ldots, n$. For each $1 \leq i \leq n$ let $\eta_i : R \to R_{f_i}$ be the canonical map. Recall that the set of contracted ideals in R along η_i, call it U_i, is in one-to-one order preserving correspondence with Ideal(R_f). Then we see that each U_i satisfies the ACC. Thus the collection of all finite intersections of ideals from any of the U_i's will satisfy the ACC.

Let $I \lhd R$ be any ideal in R. Denote by I_{f_i} the extension of I along η_i and by $I_{f_i}|_R$ the contraction of I_{f_i} along η_i. Recall that for each $x \in I_{f_i}|_R$ we have that there is a $k_i \in \mathbb{N}$ such that $xf_i^{k_i} \in I$. Let $\mathfrak{I} = \cap_{i=1}^n I_{f_i}|_R$, then for any $x \in \mathfrak{I}$ we have that there are $k_1, k_2, \ldots, k_n \in \mathbb{N}$ such that for each $1 \leq i \leq n$ we have $xf_i^{k_i} \in I$. Thus we see for each $1 \leq i \leq n$ that $f_i \in \sqrt{(I : x)}$. Since the X_{f_i}'s cover Spec(R) we must have for all prime ideals $P \lhd R$ that at least one $f_i \notin P$. Thus if all f_i's are in $\sqrt{(I : x)}$ we have that there are no prime ideals above $\sqrt{(I : x)}$. Thus $\sqrt{(I : x)} = R$ so $(I : x) = R$. This means that $x \in I$. Therefore, since $I \subseteq \mathfrak{I}$, we have shown that $I = \mathfrak{I}$. Hence every ideal in R is a finite intersection ideals from various U_i's. So we have that Ideal R satisfies ACC and hence R is Noetherian.

Let $R = \prod \mathbb{Z}_2$ be the countable product of copies of $\mathbb{Z}_2$. It is easy to see that R is not Noetherian. Note that every element of R is idempotent. Let $\mathfrak{p}$ be any prime ideal in R. Then $R_{\mathfrak{p}}$ is a local ring and for any $\frac{a}{s} \in R_{\mathfrak{p}}$ we have that $a^2 s = as^2$ so $\frac{a}{s} = (\frac{a}{s})^2$. This implies that every element of $R_{\mathfrak{p}}$ is idempotent. We proved in a previous problem that in a local ring the only idempotents are 0 and 1. Thus we must have $R_{\mathfrak{p}} \cong \mathbb{Z}_2$, which is Noetherian. Therefore we have that R is not Noetherian but $R_{\mathfrak{p}}$ is Noetherian for all prime ideals $\mathfrak{p}$ in R.