Commutative Algebra: Homework 5

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Problem 7(a): A Noetherian space is a union of finitely many $\cup$ -irreducible ($A = B \cup C \to A = B$ *or* $A = C$) closed sets.

We assume the following:

Lemma 1. *Let L be a lattice satisfying the accending chain condition. Every element of L is a finite meet of $\cap$-irreducible elements.*

Up to a change in language a topology is a lattice with lattice points corresponding to open sets. The lemma can thus be restated as follows:

Lemma 2. *Let X be a Noetherian topological space. Every open set of X is a finite intersection of $\cap$-irreducible open sets*

It follows that every closed set is a finite union of $\cup$-irreducible closed sets. As X is a closed set it too is a union of finitely many $\cup$-irreducible closed sets. □

Problem 7(b): If a ring R is Noetherian and $I \lhd R$ is an ideal, then there are finitely many minimal primes over I.

Claim 1. *The only $\cup$-irreducible closed sets in $Spec(R)$ are of the form $V(P)$, where P is a prime ideal.*

Proof. Let $F = V(E)$ be an $\cup$-irreducible closed set in $Spec(R)$. Since $V(E) = V(\sqrt{E})$, we may assume E is a radical ideal. Suppose $AB \subset E$, where A and B are ideals. Then $V(E) \subseteq V(AB) = V(A) \cup V(B)$, so

$$V(E) = (V(A) \cap V(E)) \cup (V(B) \cap V(E)).$$

Since $V(E)$ is $\cup$-irreducible, $V(E) = V(A) \cap V(E)$ or $V(E) = V(B) \cap V(E)$. We conclude that $V(E) \subseteq V(A) = V(\sqrt{A})$ or $V(E) \subseteq V(B) = V(\sqrt{B})$. Hence, $\sqrt{A} \subseteq E$ or $\sqrt{B} \subseteq E$, since for any ideal I, $V(I)$ is the set of prime ideals containing I . Since $A \subseteq \sqrt{A}$ and $B \subseteq \sqrt{B}$, we conclude that $A \subseteq E$ or $B \subseteq E$, so E is prime. □

Let I be an ideal of R, and let $\{P_\alpha | \alpha \in \lambda\}$ be the collection of minimal prime ideals above I. Since R is Noetherian, $R' = R/I$ is Noetherian, as well. Thus, $Spec(R')$ is a Noetherian topological space, so

$$Spec(R') = \bigcup_{i=1}^{n} V(Q_i),$$

where Q_i is a prime ideal in R' for all $i \leq n$. For each $\alpha \in \lambda$, let P'_α be the image of P_α in the quotient modulo I. Then for a given $\alpha \in \lambda$,

$$V(P'_\alpha) \subset \bigcup_{i=1}^{n} V(Q_i),$$

so, in particular, $P'_\alpha \in V(Q_i)$ for some $i \leq n$. Hence, $Q_i \subset P'_\alpha$, and since P'_α is a minimal prime over 0, $P_\alpha = Q_i$. Hence, λ has at most n elements. We conclude that there are only finitely many minimal primes over I. $\square$