Assignment V, Problem 6 Martinez, Tuley

6 *A topological space is* Noetherian *if it satisfies the ascending chain condition on open sets.*

(a) *Show that a space is Noetherian iff every open subspace is compact iff every subspace is compact.*

Solution. We label the above conditions (1), (2) and (3) respectively. Suppose X is a Noetherian topological space.

(1) $\Rightarrow$ (3): Suppose A is a subspace of X. Take any open cover $\{U_\alpha\}_{\alpha\in I}$ of A. Construct a nested sequence, beginning with $O_1 = U_{\alpha_1}$ for any choice of U_{α_1}. Choose $U_{\alpha_2} \notin O_1$, so that $O_2 = O_1 \cup U_{\alpha_2} \supsetneq O_1$. For each k, if $O_k \supset A$, we have a finite subcover, given by $\{U_{\alpha_i}\}_{i=1}^k$; otherwise, choose $U_{\alpha_{k+1}} \notin O_k$ and let $O_{k+1} = O_k \cup U_{\alpha_{k+1}}$. If there is no finite n such that $O_n \supset A$, then we have an infinite ascending chain, violating that X is Noetherian. Since $\{U_\alpha\}$ was an arbitrary open cover of the arbitrary subspace A, we have that *every* open cover of *every* subspace has a finite subcover, and so each subspace is compact.

(3) $\Rightarrow$ (2) trivially.

(2) $\Rightarrow$ (1): Suppose Y is a non-Noetherian space. Let $\{A_i\}$ be an infinite ascending chain of open sets, and put $A = \bigcup A_i$. Suppose by way of contradiction that $A = \bigcup_{j=1}^n A_{i_j}$; that is, A is compact. Let $m = \max i_j$; then since the set $\{A_{i_j}\}$ is nested, $A = A_m$. However, by the ascending condition on the sequence, there is some element $x \in A_{m+1} \setminus A_m$; by our earlier definition, $x \in A$ but by the compactness argument, $x \notin A$. Hence A is *not* compact. Thus if X is *not* Noetherian, there exists a non-compact open subspace. ∎

(b) *Show that if R is a Noetherian ring, then* $\operatorname{Spec}(R)$ *is Noetherian.*

Solution. The lattice of open sets in $\operatorname{Spec}(R)$ is identical to the lattice of radical ideals in R. Suppose that $\operatorname{Spec}(R)$ is *not* Noetherian; then there is an infinite ascending chain in the radical ideal lattice of R. We wish to show that this corresponds to an infinite chain in the ideal lattice of R; this is nontrivial since the radical ideal lattice is in fact a quotient of the ideal lattice.

Let $\nu : \operatorname{Ideal}(R) \to \operatorname{Ideal}(R)/\sim$ by $I \mapsto \sqrt{I}$. Suppose that $\mathfrak{r}_1 \subsetneq \mathfrak{r}_2 \subsetneq \cdots$ is a strictly increasing chain of radical ideals. We claim that $\mathfrak{i}_1 = \nu^{-1}(\mathfrak{r}_1) \subsetneq i_2 = \nu^{-1}(\mathfrak{r}_2) \subsetneq \cdots$ is a strictly increasing chain in the lattice of ideals of R. It is possible that $\mathfrak{i}_1 \notin \mathfrak{i}_2$, so let $\mathfrak{j}_2 = \mathfrak{i}_1 \vee \mathfrak{i}_2$; this ideal also has $\sqrt{\mathfrak{j}_2} = \mathfrak{r}_2$. Similarly, the pullback of $\mathfrak{r}_3$ may not contain $\mathfrak{j}_2$, so let the chain contain their join, and so on. This creates an infinite ascending chain: since $\mathfrak{r}_k \neq \mathfrak{r}_{k-1}$, there must be some element whose power is in $\mathfrak{r}_k$ but not $\mathfrak{r}_{k-1}$. So if $\operatorname{Spec}(R)$ is not Noetherian, then R is likewise not Noetherian. ∎

(c) *Give an example of a non-Noetherian ring R where* $\operatorname{Spec}(R)$ *is a Noetherian topological space.*

Solution. For a field K, let $S = K\left[x, \sqrt{x}, \sqrt{\sqrt{x}}, \dots\right]$ with the additional relation that $x^2 = 0$.[1] Define $\mathfrak{m} = \left(x, \sqrt{x}, \sqrt{\sqrt{x}}, \dots\right)$. We claim that $\mathfrak{m}$ is a maximal ideal: it contains all finite sums of elements with zero constant terms, so anything outside of $\mathfrak{m}$ contains a finite sum of elements with a nonzero constant term. Thus if we take an element $\alpha_0 + \alpha_1 x + \alpha_2\sqrt{x} + \cdots + \alpha_n \sqrt[2^n]{x}$ and add it to $\mathfrak{m}$, we may subtract from it $\alpha_1 x + \ldots + \alpha_n \sqrt[2^n]{x} \in \mathfrak{m}$ and thus obtain $\alpha_0 \in \mathfrak{m}$; however, α_0 is a unit so $\left(\mathfrak{m} \cup \left\{\alpha_0 + \cdots + \alpha_n \sqrt[2^n]{x}\right\}\right) = S$.

Since $\mathfrak{m}$ is maximal, it is prime and hence $S \setminus \mathfrak{m}$ is multiplicatively closed. Let $R = S_{\mathfrak{m}}$. In R, everything outside of $\mathfrak{m}$ is a unit. Furthermore, $\mathfrak{m}$ is the radical of every ideal below $\mathfrak{m}$: since $\sqrt{\mathfrak{m}} = \mathfrak{m} = \sqrt{(0)}$, it is the radical of every ideal in the interval between (0) and $\mathfrak{m}$.

We also claim that R is not Noetherian: Noetherian rings have finitely-generated nilradicals, *i.e.* every element has a bounded nilpotence degree. However, by construction, R has elements whose nilpotence degree is 2^k for all $k \in \mathbb{N}$.

Observe that $\mathfrak{m} = \operatorname{Nil}(R)$, so $\mathfrak{m}$ is contained in every prime but is also maximal, so it is the *only* prime in R, hence $\operatorname{Spec}(R)$ consists of a single point, $\mathfrak{m}/\sim$, and so is Noetherian. ∎

[1] A note on the construction of S: ideals $(a_1) \subset (a_2) \subset (a_3) \subset \cdots$ form an ascending chain when $a_2 \mid a_1$, $a_3 \mid a_2$, and so on. Additionally, we need that $\mathfrak{m}$ is the radical of all of the ideals below it, including the ideal (0), so we have to have that all elements are nilpotent.