Math 6150, Assignment 5, Problem 5

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#5. Show in some sense that the weak form of Nakayama's lemma is equivalent to the statement (†) "If R is a local ring and M and N are finitely generated R-modules, then $M \otimes_R N = 0$ implies $M = 0$ or $N = 0$."

Proof. We first show that (†) implies Nakayama's lemma. Let R be any commutative ring. Assume I is an ideal contained in $\mathrm{rad}(R)$ and let M be a finitely generated R-module that satisfies $M = IM$. We have to show $M = 0$. Begin with the exact sequence

$$0 \to IM \to M \to {}^{M}\!/_{IM} \to 0.$$

We then localize along a to-be-specified set S. Localization is exact, so the resulting sequence

$$0 \to (IM)_S \to M_S \to ({}^{M}\!/_{IM})_S \to 0$$

is exact as well. We now have R_S-modules in the picture. We need tensor products of them if we want to apply (†). So we'll tensor this sequence with a well-chosen R_S-module. Let $\mathfrak{m}$ be the maximal ideal of R_S and let $\mathbb{F} = {}^{R_S}\!/_{\mathfrak{m}R_S}$ denote the residue field. $\mathbb{F}$ is a simple R_S-module. Apply the functor $(\cdot) \otimes_{R_S} \mathbb{F}$ to our sequence. The resulting sequence will be right exact. Specifically, it looks like

$$(IM)_S \otimes_{R_S} \mathbb{F} \longrightarrow M_S \otimes_{R_S} \mathbb{F} \longrightarrow ({}^{M}\!/_{IM})_S \otimes_{R_S} \mathbb{F} \longrightarrow 0.$$

Now apply what we know about these modules. Since $M = IM$, the quotient ${}^{M}\!/_{IM}$ is trivial, so the third term of this sequence is trivial. We claim the first term in this sequence is also trivial if S is chosen judiciously. Suppose S is the complement of a prime containing I, so that elements of I aren't inverted when passing to the localization R_S. As nonunits, elements of I must reside in the maximal ideal $\mathfrak{m}$ of R_S. Since R_S is local, its maximal ideal $\mathfrak{m}$ is its Jacobson radical. The Jacobson radical of R_S consists of elements that annihilate all simple R_S-modules; in particular we see that I will annihilate $\mathbb{F}$. So given a tensor $\frac{im}{s} \otimes f \in (IM)_S \otimes_{R_S} \mathbb{F}$, we may bring i to the other side (considering $i = \frac{i}{1}$ as a scalar in R_S):

$$\frac{im}{s} \otimes f = \frac{m}{s} \otimes if = \frac{m}{s} \otimes 0 = 0,$$

since i annihilates f. This transfer of i does not cause the first factor to leave $(IM)_S$, as $M = IM$ we know that $\frac{m}{s}$ is an element of $(IM)_S$ in disguise. The centered equation shows that every tensor is zero and the module is zero. Now $M_S \otimes_{R_S} \mathbb{F}$ is sandwiched in an exact sequence between two trivial modules; it must be trivial as well. Apply (†); it says either $M_S = 0$ or $\mathbb{F} = 0$. The latter is absurd, so we conclude $M_S = 0$, and therefore $M = 0$ as desired. We only need to know that there is a prime containing I, and since I is by hypothesis a proper ideal, these exist (in fact, we know there are even minimal ones).

For the other direction, suppose that neither $N = 0$ or $M = 0$. We will show that for a local ring R, $M \otimes_R N$ is non-trivial. If both M and N are non-trivial, then the contrapositive of Nakayama's Lemma implies that both $\mathfrak{m}M \neq M$ and $\mathfrak{m}N \neq N$. Therefore both ${}^{M}\!/_{\mathfrak{m}M}$ and ${}^{N}\!/_{\mathfrak{m}N}$ are non-trivial, and by ca3p6(a), are both vector spaces over ${}^{R}\!/_{\mathfrak{m}}$. Therefore, ${}^{M}\!/_{\mathfrak{m}M} \otimes_{R/\mathfrak{m}} {}^{N}\!/_{\mathfrak{m}N}$ is a nontrivial ${}^{R}\!/_{\mathfrak{m}}$ vector space. We may also consider ${}^{M}\!/_{\mathfrak{m}M} \otimes_{R/\mathfrak{m}} {}^{N}\!/_{\mathfrak{m}N}$ as an R-module, where action by an element of R is given by the action of its coset in ${}^{R}\!/_{\mathfrak{m}}$. Therefore we need only exhibit a non-trivial R-module homomorphism

$$h : M \otimes_R N \longrightarrow {}^{M}\!/_{\mathfrak{m}M} \otimes_{R/\mathfrak{m}} {}^{N}\!/_{\mathfrak{m}N}$$

to show that $M \otimes_R N$ is non-trivial. This is the same as showing that the map

$$f : M \times N \to {}^{M}\!/_{\mathfrak{m}M} \otimes_{R/\mathfrak{m}} {}^{N}\!/_{\mathfrak{m}N} \text{ given by } f(m, n) = (m + \mathfrak{m}M) \otimes_{R/\mathfrak{m}} (n + \mathfrak{m}N)$$

is an R-bilinear map. f is obviously linear in each coordinate. Furthermore, f respects the tensor product, as

$$\begin{aligned} f(rm, n) &= (rm + \mathfrak{m}M) \otimes_{R/\mathfrak{m}} (n + \mathfrak{m}N) \\ &= ((r + \mathfrak{m})m + \mathfrak{m}M) \otimes_{R/\mathfrak{m}} (n + \mathfrak{m}N) \\ &= (m + \mathfrak{m}M) \otimes_{R/\mathfrak{m}} ((r + \mathfrak{m})n + \mathfrak{m}N) \\ &= (m + \mathfrak{m}M) \otimes_{R/\mathfrak{m}} (rn + \mathfrak{m}N). \end{aligned}$$

f is clearly non-trivial, therefore it extends to a non-trivial R-module homomorphism. Therefore, $M \otimes_R N$ is non-trivial, as desired.

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