COMMUTATIVE ALGEBRA HOMEWORK V

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Problem 4

Let $L, N \leq M$ be R-modules. Let U be the set of primes $\mathfrak{p}$ for which $L_\mathfrak{p} \subseteq N_\mathfrak{p}$ holds. Show that U is an intersection of open sets in $\mathrm{Spec}(R)$. Show conversely that if V is any intersection of open sets in $\mathrm{Spec}(R)$, then V is exactly the set of primes for which $L_\mathfrak{p} \subseteq N_\mathfrak{p}$ holds for some submodules L, N of some module M.

Solution Let $L, N \leq M$ be R-modules. Let U be the set of primes $\mathfrak{p}$ for which $L_\mathfrak{p} \subseteq N_\mathfrak{p}$ holds. We wish to show that U is an intersection of open sets, but it is equivalent to show that

$$U^C = \{\mathfrak{p} \mid \mathfrak{p} \text{ is prime and } L_\mathfrak{p} \nsubseteq N_\mathfrak{p}\}$$

is a union of closed sets. To this end we show that if $\mathfrak{p} \in U^C$ and $\mathfrak{p} \subseteq \mathfrak{q}$ for some prime $\mathfrak{q}$ then $\mathfrak{q} \in U^C$. Recall that for an R-module M and prime ideal $\mathfrak{p} \lhd R$ we define $M_\mathfrak{p}$ to be the R-module of equivalence classes with representatives m/s where $m \in M$ and s is an element of the multiplicatively closed set $S(\mathfrak{p}) = R - \mathfrak{p}$. The equivalence relation is given by $m/s \sim m'/s'$ if there exists $t \in S(\mathfrak{p})$ such that $t(ms' - m's) = 0$. Let $\mathfrak{p} \in U^C$ and suppose $\mathfrak{q}$ is a prime above $\mathfrak{p}$. Then by the definition of U^C we have $L_\mathfrak{p} \nsubseteq N_\mathfrak{p}$ so there an element of $L_\mathfrak{p}$ that is not in $N_\mathfrak{p}$. That is, there exists $\ell \in L$, $s \in S(\mathfrak{p})$ such that if $n \in N$ and $t, s' \in S(\mathfrak{p})$ we have

$$t(\ell s' - ns) \neq 0. \qquad (*)$$

Let $\mathfrak{q}$ be a prime above p. Since we do not consider R to be prime, $S(\mathfrak{q})$ is nonempty. Let $x \in S(\mathfrak{q})$. We claim that $\ell/x \in L_\mathfrak{q}$ is not equal to any element of $N_\mathfrak{q}$. Suppose not. Then there exists $n \in N$ and $t, s' \in S(\mathfrak{q})$ such that

$$t(\ell s' - nx) = 0.$$

Since $S(\mathfrak{q}) \subseteq S(\mathfrak{p})$ are multiplicatively closed sets we have $t, xs, s' \in S(\mathfrak{p})$. Then in $N_\mathfrak{p}$ we have $\ell s/xs = \ell/x = n/s'$, that is there exists $t' \in S(\mathfrak{p})$ such that

$$t'(\ell(ss') - (nx)s) = 0.$$

But $ss' \in S(\mathfrak{p})$ and $nx \in N$, contradicting (*). It follows that $L_\mathfrak{q} \nsubseteq N_\mathfrak{q}$ and hence $\mathfrak{q} \in U^C$. Since $\mathfrak{q}$ was an arbitrary prime above $\mathfrak{p}$ it follows that the collection

$$F_\mathfrak{p} = \{\mathfrak{q} \mid \mathfrak{q} \text{ is a prime above } \mathfrak{p}\}$$

is a subset of U^C whenever $\mathfrak{p} \in U^C$. Furthermore, $F_\mathfrak{p}$ is closed, by the definition of the topology on $\mathrm{Spec}(R)$. It is now clear that

$$U^C = \bigcup_{p \in U^C} F_\mathfrak{p}$$

and hence that U^C is a union of closed sets. Then U is an intersection of open sets as required.

We now wish to show that if V is any intersection of open sets in Spec(R), then V is exactly the set of primes for which $L_\mathfrak{p} \subseteq N_\mathfrak{p}$ holds for some submodules L, N of some submodule M. It is equivalent to show that V^C is precisely the set of primes for which $L_\mathfrak{p} \nsubseteq N_\mathfrak{p}$. Since V is an intersection of open sets, V^C is a union of closed sets. From the definition of the topology on Spec(R), we have

$$V^C = \bigcup_\beta V(E_\beta)$$

where the sets E_β are subsets of R and

$$V(E_\beta) = \{\mathfrak{p} \mid \mathfrak{p} \text{ is prime and contains } E_\beta\}.$$

But if a prime $\mathfrak{p}$ contains the subset E_β, it contains the ideal $I_\beta = (E_\beta)$ so we have

$$V^C = \bigcup_\beta V(I_\beta).$$

It has been shown previously [see ca2p5] that if the ideal I is contained in the prime ideal $\mathfrak{p}$ then there is a minimal prime $\mathfrak{q}$ such that $I \subseteq q \subseteq p$. Then for every prime in $\mathfrak{p}$ in V^C and every β there exists a prime ideal q such that $I_\beta \subseteq \mathfrak{q} \subseteq \mathfrak{p}$. Then WOLG we may express the above union as

$$V^C = \bigcup_\alpha V(\mathfrak{q}_\alpha)$$

where $\mathfrak{q}_\alpha$ is prime for all α.

Let M be the free R-module with free basis $\{e_\alpha\}$. That is, each element of M has the form $\sum_\alpha r_\alpha e_\alpha$ where only a finite number of $r_\alpha \in R$ are nonzero. Take $N \leq M$ to be the collection of all elements in M of the form $\sum_\alpha a_\alpha e_\alpha$ where $a_\alpha \in \mathfrak{q}_\alpha$ for all α. We will show that

$$V^C = \bigcup_\alpha V(\mathfrak{q}_\alpha) = \{\mathfrak{p} \mid \mathfrak{p} \text{ is prime and } M_\mathfrak{p} \nsubseteq N_\mathfrak{p}\}$$

Suppose that $\mathfrak{p} \in V(\mathfrak{q}_\lambda)$ for some λ. Then $\mathfrak{q}_\lambda \subseteq \mathfrak{p}$. Consider $1 \cdot e_\lambda \in M - N$. Since $1 \in S(\mathfrak{p}) = R - \mathfrak{p}$, we have that $(1 \cdot e_\lambda)/1 \in M_\mathfrak{p}$ and claim that $(1 \cdot e_\lambda)/1 \notin N_\mathfrak{p}$. Suppose to the contrary. Then $(1 \cdot e_\lambda)/1 = n/s$ for some $n \in N$ and some $s \in S(\mathfrak{p})$. Then there exists $t \in S(\mathfrak{p})$ such that

$$t(se_\lambda - n) = 0.$$

Hence, $tse_\lambda = tn \in N$. Then we must have $ts \in \mathfrak{q}_\lambda$ by the definition of N. But $t, s \in S(\mathfrak{p})$ and $S(\mathfrak{p})$ is multiplicatively closed, so $ts \in S(\mathfrak{p}) = R - \mathfrak{p} \subseteq R - \mathfrak{q}_\lambda$. That is, $ts \notin \mathfrak{q}_\lambda$, a contradiction. Thus our supposition was false and we have $e_\lambda/1 \notin N$, which implies $M_\mathfrak{p} \nsubseteq N_\mathfrak{p}$. Hence, we have $V^C \subseteq \{\mathfrak{p} \mid \mathfrak{p} \text{ is prime and } M_\mathfrak{p} \nsubseteq N_\mathfrak{p}\}$.

On the other hand suppose $\mathfrak{p} \notin V(\mathfrak{q}_\alpha)$ for all α. Then $\mathfrak{q}_\alpha \nsubseteq \mathfrak{p}$. For each α, choose $a_\alpha \in \mathfrak{q}_\alpha - \mathfrak{p} = \mathfrak{q}_\alpha \cap S(\mathfrak{p})$. Then we have $(1 \cdot e_\alpha)/1 = (a_\alpha \cdot e_\alpha)/a_\alpha \in N_\mathfrak{p}$. Since $M_\mathfrak{p}$ is generated as an $R_\mathfrak{p}$ module by the elements $(1 \cdot e_\alpha)/1$, it follows that $M_\mathfrak{p} \subseteq N_\mathfrak{p}$. Hence we have $V^C \supseteq \{\mathfrak{p} \mid \mathfrak{p} \text{ is prime and } M_\mathfrak{p} \nsubseteq N_\mathfrak{p}\}$. Therefore

$$V^C = \{\mathfrak{p} \mid \mathfrak{p} \text{ is prime and } M_\mathfrak{p} \nsubseteq N_\mathfrak{p}\},$$

and

$$V = \{\mathfrak{p} \mid \mathfrak{p} \text{ is prime and } M_\mathfrak{p} \subseteq N_\mathfrak{p}\}.$$

The result follows. $\square$