COMMUTATIVE ALGEBRA: HOMEWORK 5

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3) Let $N \leq M$ be R-module, $I \lhd R$ an ideal of R, and $m, n \in M$ and $r \in R$.

(a) Show that the set of primes $\mathfrak{p}$ where "$m \in N_\mathfrak{p}$" is an open subset of $\text{Spec}(R)$ and that it is all of $\text{Spec}(R)$ if and only if $m \in N$.

(b) Same type of problem for "$m = 0$ in $M_\mathfrak{p}$".

(c) Same type of problem for "$m = n$ in $M_\mathfrak{p}$".

(d) Same type of problem for "r is nilpotent in $R_\mathfrak{p}$'.

(e) Same type of problem for "r is a unit in $R_\mathfrak{p}$".

Solution

(a) *Proof.* $m \in N_\mathfrak{p}$ if and only if there is $a \in N_\mathfrak{p}$ such that $m = a$, which is true if and only if there is $u \in R \setminus \mathfrak{p}$ such that $um = ua$ in N. Therefore,

$$V = \{\mathfrak{p} \mid m \in N_\mathfrak{p}\} = \{\mathfrak{p} \mid \exists u \notin \mathfrak{p}\ um \in N\}.$$

Let $U = (N : m)$ and $A = \{\mathfrak{p} \mid U \not\subseteq \mathfrak{p}\}$. Suppose that $\mathfrak{p} \in A$. Then $U \not\subseteq \mathfrak{p}$, so there is $u \in U \setminus \mathfrak{p}$ such that $um \in N$. Hence $\mathfrak{p} \in V$. Suppose that $\mathfrak{p} \in V$. Then there is $u \in R \setminus \mathfrak{p}$ such that $um \in N$. Hence $u \in U$, so $U \not\subseteq \mathfrak{p}$. Therefore $\mathfrak{p} \in A$, so $V = A$. Since A is open, V is also open.

Suppose that $m \in N$. Then $1 \notin \mathfrak{p}$ and $1m \in N$ for every prime ideal $\mathfrak{p}$. Thus $V = \text{Spec}(R)$. Conversely, if $V = \text{Spec}(R)$ then the set U above is not contained in any prime ideal. However, if U is proper then it is contained in some maximal ideal. Maximal ideals are prime, so this is a contradiction. Hence $U = R$, so in particular $1 \in U$ and $1m = m \in N$. Therefore $V = \text{Spec}(R)$ if and only if $m \in N$. □

(b) *Proof.* $(0)_\mathfrak{p}$ is the submodule of $M_\mathfrak{p}$ generated by 0. Thus, $m = 0$ in $M_\mathfrak{p}$ if and only if $m \in (0)_\mathfrak{p}$. By part (a) above, the set

$$V = \{\mathfrak{p} \mid m = 0 \text{ in } M_\mathfrak{p}\} = \{\mathfrak{p} \mid m \in (0)_\mathfrak{p}\}.$$

is open and $V = \text{Spec}(R)$ if and only if $m \in (0)$. $m \in (0)$ if and only if $m = 0$, so $V = \text{Spec}(R)$ if and only if $m = 0$. □

(c) *Proof.* $m = n$ in $M_\mathfrak{p}$ if and only if $m - n = 0$ in $M_\mathfrak{p}$. Using part (b) above, we have that

$$V = \{\mathfrak{p} \mid m = n \text{ in } M_\mathfrak{p}\} = \{\mathfrak{p} \mid m - n = 0 \text{ in } M_\mathfrak{p}\}$$

is open and that it is all of $\text{Spec}(R)$ if and only if $m - n = 0$ in M. Hence $V = \text{Spec}(R)$ if and only if $m = n$ in M. □

(d) *Proof.* r is nilpotent in $R_\mathfrak{p}$ if and only if there is $n \in \mathbb{N}$ such that $r^n = 0$ in $R_\mathfrak{p}$. By part (b) above, each set $U_n = \{\mathfrak{p} \mid r^n = 0 \text{ in } R_\mathfrak{p}\}$ is open. Hence

$$V = \{\mathfrak{p} \mid r \text{ nilpotent in } R_\mathfrak{p}\} = \bigcup_{n \in \mathbb{N}} U_n$$

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is open.

Suppose that $V = \operatorname{Spec}(R)$. If $r^n = 0$ in $R_{\mathfrak{p}}$ then $r^{n+1} = 0$ in $R_{\mathfrak{p}}$. Therefore $U_n \subseteq U_{n+1}$. Since $\operatorname{Spec}(R)$ is compact and $\bigcup U_n = \operatorname{Spec}(R)$, this implies that there must be some m such that $U_m = \operatorname{Spec}(R)$. By the second part of part (b), this is true if and only if $r^m = 0$ in R. Hence r is nilpotent in R. Suppose that r is nilpotent. Then there is $n \in \mathbb{N}$ such that $r^n = 0$ in R, so $U_n = \operatorname{Spec}(R)$ and hence $V = \operatorname{Spec}(R)$. Therefore $V = \operatorname{Spec}(R)$ if and only if r is nilpotent. $\square$

(e) *Proof.* $r \in R_{\mathfrak{p}}$ is a unit if and only if there is $s/t \in R_{\mathfrak{p}}$, with $s \in R$ and $t \notin \mathfrak{p}$ such that $r(s/t) = 1$. This is true if and only if $rs = t$ in $R_{\mathfrak{p}}$. Therefore

$$V = \{\mathfrak{p} \mid r \text{ is a unit in } R_{\mathfrak{p}}\} = \{\mathfrak{p} \mid \exists s \in R\, \exists t \notin \mathfrak{p}\ rs = t \text{ in } R_{\mathfrak{p}}\}.$$

Let

$$A_{s,t} = \{\mathfrak{p} \mid rs = t \text{ in } R_{\mathfrak{p}}\} \cap \{\mathfrak{p} \mid t \notin \mathfrak{p}\}.$$

By part (c) above, this is the intersection of two open sets. Therefore it is open. Let $A = \bigcup_{s,t \in R} A_{s,t}$. Suppose that $\mathfrak{p} \in A$. Then there are $s, t \in R$ such that $p \in A_{s,t}$. Hence $sr = t$ in $R_{\mathfrak{p}}$ and $t \notin \mathfrak{p}$. Therefore $r(s/t) = 1$ in $R_{\mathfrak{p}}$, so $\mathfrak{p} \in V$. Suppose that $\mathfrak{p} \in V$. Then there are $s, t \notin \mathfrak{p}$ such that $rs = t$ in $R_{\mathfrak{p}}$. Hence $\mathfrak{p} \in A_{s,t}$, so $\mathfrak{p} \in A$. Therefore $\mathfrak{p} \in A$, and $A = V$. Since A is open, V is also open.

Suppose that r is not a unit in R. Then r is contained in some maximal (and thus prime) ideal $\mathfrak{p}$. Now, r is a unit in $R_{\mathfrak{p}}$ if and only if there are $s, t, u \notin \mathfrak{p}$ such that $u(rs - t) = 0$. Therefore $urs = ut$. Since $u, t \notin \mathfrak{p}$, $ut \notin \mathfrak{p}$. Since $r \in \mathfrak{p}$, $urs \in \mathfrak{p}$. This is a contradiction, so r cannot be a unit in $R_{\mathfrak{p}}$. Therefore $\mathfrak{p} \notin V$, so $V \neq \operatorname{Spec}(R)$. Suppose that r is a unit in R. Then $r^{-1}, 1 \notin \mathfrak{p}$ for all primes $\mathfrak{p}$ and $1(r^{-1}r - 1) = 0$. Hence $V = \operatorname{Spec}(R)$. Therefore $V = \operatorname{Spec}(R)$ if and only if r is a unit in R. $\square$