Assignment V

Commutative Algebra, Math 6150

Kearnes 12:00

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Problem #2

Let R be a commutative ring. The following are equivalent.

(a) $R_{\mathfrak{p}}$ is a field for every prime ideal $\mathfrak{p} \lhd R$.

(b) $R_{\mathfrak{m}}$ is a field for every maximal ideal $\mathfrak{m} \lhd R$.

(c) R is a regular ring. Moreover, if R is an integral domain, then R is a field.

Proof.
(a) implies (b): Every maximal ideal is a prime ideal, so this implication is trivial.

(b) implies (c): Let $I \lhd R$ be contained in some maximal ideal $\mathfrak{m}$. Let $a \in I$ and examine $\frac{a}{1} \in R_{\mathfrak{m}}$. Because $R_{\mathfrak{m}}$ is a field, we know that $\frac{a}{1}$ is either a unit or zero.

If $\frac{a}{1}$ is a unit, then there is some $\frac{b}{s} \in R_{\mathfrak{m}}$, with $b \in R$ and $s \in R \setminus \mathfrak{m}$, such that $\frac{a}{1}\frac{b}{s} = \frac{1}{1}$, i.e. there exists $u \in R \setminus \mathfrak{m}$ such that $(ab1 - 1s1)u = 0$. A little algebra gives us that $abu = su$. Now, $abu \in I$ because I is an ideal and $su \in R \setminus \mathfrak{m}$ because $R \setminus \mathfrak{m}$ is multiplicatively closed, yet we assumed initially that $I \subset \mathfrak{m}$, so this is a contradiction.

Therefore $\frac{a}{1}$ is zero, i.e. there is a $u \in R \setminus \mathfrak{m}$ such that $au = 0$. If I is (0), then I is idempotent clearly, so we can assume a is not zero, i.e. a is a zero divisor. We first consider the case where $I = (a)$.

Now let $A = \mathrm{Ann}(a)$. Consider the ideal $(a) + A$. If this is a proper ideal, then there is some maximal ideal $\mathfrak{n}$ containing $(a) + A$. We look at $\frac{a}{1}$ in $R_{\mathfrak{n}}$ and use the same argument to find a $u' \in R \setminus \mathfrak{n}$ such that $au' = 0$. However, this means $u' \in A \subseteq \mathfrak{n}$, so we have a contradiction.

Thus $(a) + A = R$. Since $1 \in R$, we can find $r \in R$ and $v \in A$ such that $ra + v = 1$. Now we multiply both sides by a to get $ra^2 + av = a$ or, since $av = 0$, this becomes $ra^2 = a$. We have shown that $(a) \subseteq (a^2) = (a)^2$. As the reverse inclusion is clear, we have that $I = (a)$ is idempotent.

Now assume that I is has multiple generators, $a_1, a_2, \ldots, a_n$. By the argument above, we know the principal ideals $(a_1), (a_2), \ldots, (a_n)$ are all idempotent. By Problem 7 of assignment one, we know that we can view these ideals as generated by idempotent elements, say $e_1, e_2, \ldots, e_n$. Because elements of I are sums of elements from these principal ideals, we get $I = (e_1, e_2, \ldots, e_n)$. Finally, I^2 is generated by products of pairs of generators of I, so, because $e_i^2 = e_i$ for each i, we know that I^2 contains all the generators of I and thus I itself. That is, I is idempotent.

The case when R is an integral domain: If R is an integral domain, then R has no zero divisors. We saw above that, if there is a maximal ideal $\mathfrak{m}$ containing a nonzero element a, condition (b) requires that a is a zero divisor, an impossibility. Thus there is no nontrivial maximal ideal of R. Thus $R/(0) = R$ is a field.

(c) implies (a): Suppose that every finitely generated ideal is idempotent and let $\mathfrak{p}$ be some prime ideal. Assume, for contradiction, that $R_{\mathfrak{p}}$ is not a field. Thus its unique maximal ideal M is nontrivial. So there must be a nontrivial $a \in \mathfrak{p}$ with image $\frac{a}{1}$ in $R_{\mathfrak{p}}$ nontrivial.

Consider the ideal $(a) \subseteq \mathfrak{p}$. By our assumption and problem 7 of homework 1, we know that (a) is generated by an idempotent element e. We then notice that $\frac{e}{1}$ is not zero, else $\frac{a}{1}$ would be. We also cannot have $\frac{e}{1}$ equal to a unit, else it would generate all of $R_{\mathfrak{p}}$, a contradiction to $(\frac{e}{1}) \subseteq M \subsetneq R_{\mathfrak{p}}$. Additionally, note that $\frac{e}{1}$ is idempotent since e is and $\frac{e}{1}$ is the image of a homomorphism.

So we have found an element $\frac{e}{1}$ of M that is a nonzero, nonunit idempotent. Consider the element $\frac{1}{1} - \frac{e}{1}$. This element is also nonzero, else $\frac{e}{1}$ would be a unit. If we multiply this element and $\frac{e}{1}$, we get

$$\left(\frac{1}{1} - \frac{e}{1}\right)\frac{e}{1} = \frac{e}{1} - \frac{e^2}{1} = \frac{e}{1} - \frac{e}{1} = \frac{0}{1},$$

so the element is a zero divisor and thus not a unit. As a nonzero, nonunit element, the ideal generated by this element is nontrivial and contained in a maximal ideal. Since $\mathfrak{p}$ is the unique maximal ideal, we get $(\frac{1}{1} - \frac{e}{1}) \in \mathfrak{p}$.

However, we also know that $\frac{e}{1} \in \mathfrak{p}$, so the sum

$$\frac{1}{1} - \frac{e}{1} + \frac{e}{1} = \frac{1}{1}$$

is in $\mathfrak{p}$. This is clearly a contradiction, since $\mathfrak{p}$ is proper. Therefore our assumption that $R_{\mathfrak{p}}$ is not a field is incorrect. As the choice of $\mathfrak{p}$ was arbitrary, we get that $R_{\mathfrak{p}}$ is a field for all prime ideals $\mathfrak{p}$ of R.

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