Commutative Algebra Homework 5

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1) Let R be a PID and K be its field of fractions.

a) Show that any intermediate subring $R \subseteq S \subseteq K$ is a localization of R, and is also a PID.

b) Suppose now that R is also a local ring. Show that the only intermediate rings $R \subseteq S \subseteq K$ are $S = R$ and $S = K$.

c) Show that parts (a) and (b) are false for UFDs in place of PIDs.

Solution

a) Let $R \subseteq S \subseteq K$. Let $T = \{r \in R : \frac{1}{r} \in S\}$. We claim that $S = R_T$. Let $\frac{a}{b} \in S$ be in lowest terms. Since R is a PID and a and b are coprime, the ideal $(a, b) = (1)$ in R. So there exist $x, y \in R$ such that $ax + by = 1$. Then in K, $\frac{a}{b}x + y = \frac{1}{b}$. Since $\frac{a}{b}, x, y \in S$, $\frac{1}{b} \in S$. Since $b \in R$ and $\frac{1}{b} \in S$, $b \in T$. Therefore $\frac{1}{b} \in T^{-1}$ and $\frac{a}{b} = a \cdot \frac{1}{b} \in T^{-1}R$. Thus $S \subseteq R_T$.
Now let $c \in R_T$. We know that $R_T = T^{-1}R$, so then there exist $a \in R$ and $b \in T$ such that $c = a \cdot \frac{1}{b}$. Now we see $b \in T$ implies that $\frac{1}{b} \in S$. Since $a \in R \subset S$, then $a \cdot \frac{1}{b} \in S$. Therefore $R_T \subseteq S$. Since $S \subseteq R_T$, $S = R_T$ and any intermediate subring S of R is a localization of R.
We now show that $S = R_T$ is a PID. Let $J \triangleleft R_T$ be any ideal of R_T. There exists $I \triangleleft R$ such that $J = IR_T$. R is a PID so there exists $r \in R$ such that $(r) = I$. Let $j \in J$. Since $J = IR_T$, $j = k \cdot r \cdot \frac{a}{b}$ where $k \cdot r \in I$ and $\frac{a}{b} \in R_T$. Therefore $j = k \cdot \frac{a}{b} \cdot r \in (r)$ where (r) is considered as a ideal in R_T. Thus $J \subseteq (r)$.
Now let $j \in (r)$. Then $j = \frac{a}{b}r$ where $\frac{a}{b} \in R_T$ and $r \in I$. Therefore $j \in IR_T = J$ and $(r) \subseteq J$. Thus $(r) = J$ and any ideal J is principal.

b) Suppose that R is a local ring and that $R \subseteq S \subseteq K$. Since R is a local ring, let $M \triangleleft R$ be the unique maximal ideal. By part (a), $S = R_T$ is a PID for some $T \subseteq R$. By theorem 1.3, there exists a maximal ideal $N \triangleleft R_T$. Assume that $N \neq (0)$. Since R_T is a PID and N is maximal, N is a prime ideal. Therefore N^c is prime and disjoint from T. Thus $N^c = M$. M contains all of the nonunits in R and $T \cap M = \emptyset$. Therefore T contains only units and $S = R_T = R$. If $N = (0)$, then R_T is a field. Therefore $R_T = K$.

c) Let $R = \mathbb{Z}[x]$. We easily see that R is a UFD, and not a PID. Let $S = R\left[\frac{2}{x}\right]$. Then $R \subseteq S \subseteq K$ as desired. Suppose towards contradiction there exists a T such that $S = R_T = T^{-1}R$. Without loss of generality we may assume that T is saturated since $R_T = R_{\text{sat}(T)}$. Now let $\frac{2}{x} = \frac{r_1}{t_1} \cdot \frac{r_2}{t_2}$ be a factorization of $\frac{2}{x}$ in R_T. Then $2t_1t_2 = xr_1r_2$ in $\mathbb{Z}[x]$, and $x \not| 2$ so x divides t_1 or t_2. Suppose without loss of generality that $x | t_1$. Then there is some r such that $xr = t_1$, so since T is saturated, then $r, x \in T$. Then

$\frac{1}{x} \in T^{-1} \subset R_T = S$, so

$$\frac{1}{x} = a_0(x) + a_1(x)\left(\frac{2}{x}\right) + a_2(x)\left(\frac{2}{x}\right)^2 + \cdots + a_n(x)\left(\frac{2}{x}\right)^n$$

Multiplying by x on both sides give us

$$1 = xa_0(x) + 2a_1(x) + \frac{2^2 a_2(x)}{x} + \cdots + \frac{2^n a_n(x)}{x^{n-1}}.$$

Then since $xa_0(x)$ has no constant term, we see that the constant term of the right hand side must be even, thus cannot be equal to 1, a contradiction. Then there does not exist a T such that $S = R_T = T^{-1}R$, thus S is not a localization of R, so part (a) is false if the condition of PID is weakened to UFD.

Now let $R = \mathbb{Q}[x,y]_{(x,y)}$, and let K be its field of fractions. We see that any element of $R \setminus (x,y)$ is a unit, so (x,y) is the unique maximal ideal. Therefore R is a local ring. R is a clearly a UFD. Now assume that R is a PID. Then we see that $(x),(y),(x,y)$ are all nonzero prime ideals, and are thus maximal since R is a PID. Then $(x),(y) \subset (x,y)$, so $(x) = (x,y) = (y)$, thus x and y are associates in R. Then there is some unit $u \in R$ such that $uy = x$. Then we must have that $u = \frac{x}{y}$ is a unit in R. Then since $\frac{x}{y} \in R$ we have that $\frac{x}{y} = \frac{f(x,y)}{g(x,y)}$ for some $f \in \mathbb{Q}[x,y]$ and $g \in \mathbb{Q}[x.y] \setminus (x,y)$. This implies that $yf = gx$, but for this to happen y must divide g, contradicting $g \notin (x,y)$. Therefore R is not a PID.

Now consider $R_{(y)}$. Then we see that $x \in R \setminus (y)$, so $\frac{1}{x} \in R_{(y)}$, but $\frac{1}{x} \notin R$, so $R \subsetneq R_{(y)}$. Now we see that $\frac{1}{y} \notin \mathbb{Q}[x,y]$, and $y \in (x,y)$, so $\frac{1}{y} \notin \mathbb{Q}[x,y]_{(x,y)} = R$. Also, $y \in (y)$, so $\frac{1}{y} \notin R_{(y)}$, thus $R_{(y)} \subsetneq K$. Thus, part (b) is false if the condition of PID is weakened to UFD.