COMMUTATIVE ALGEBRA: HOMEWORK 4

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(9) Show that if M is free, projective, flat, finitely generated, or finitely presentable as an R-module, then so is M_S as an R_S-module.

Solution

Proof. We first show that if X is a generating set for M as an R-module, then the set $X_S = \{\frac{x}{1} : x \in X\}$ generates M_S. Let $\frac{m}{s}$ be an element of M_S. Then as X generates M, we have that for some finite subset I that $m = \sum_{i \in I} r_i x_i$. Then it follows that

$$\frac{m}{s} = \sum_{i \in I} (\frac{r_i}{s})(\frac{x_i}{1})$$

hence X_S is a generating set. Therefore if M is finitely generated, then M_S is also finitely generated.

Now, suppose that X freely generates M. To show that M_S is free it suffices to show that X_S is R_S linearly independent. Let I be a finite indexing set, and suppose that we have that

$$\sum_{i \in I} (\frac{r_i}{s_i})(\frac{x_i}{1}) = 0$$

As the set S is multiplicatively closed, we can find a common denominator and rewrite the above equation as

$$\frac{\sum_{i \in I} r'_i x_i}{s'} = 0$$

Therefore, there exists a $u \in S$ such that $u \sum_{i \in I} r'_i x_i = 0$. Distributing the u inside the sum, and using the linear independence of X we see that for all $i \in I$ $ur'_i = 0$. Finally, we see that

$$\frac{r_i}{s_i} = \frac{r'_i}{s'} = \frac{ur'_i}{us'} = 0$$

Therefore, all of the coefficients in our original dependence relation are zero, so X_S freely generates M_S.

Now suppose that M is projective as an R-module. Then M is a direct summand of a free R-module, call it F. Then we have $F \cong N \oplus M$. We recall that $M_S \cong M \otimes_R R_S$. Then $F_S \cong (N \oplus M) \otimes_R R_S$. Using that tensoring distributes over a direct sum, and that localizing free modules produces free modules, we have

$$F_S \cong (N \otimes_R R_S) \oplus (M \otimes_R R_S) \cong N_S \oplus M_S$$

and hence M_S is projective.

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We now show that if M is flat as an R-module, then M_S is flat as an R_S-module. Recall that for any ring

$$\varphi : R \to R_S : r \mapsto \frac{r}{1}$$

is a homomorphism of R into its localization with respect to S. This map can be used to restrict the scalars of R_S-modules to make them R-modules. Now suppose that

$$0 \to A \to B \to C \to 0$$

is an exact sequence of R_S-modules. Using that any R-module M is isomorphic to $M \otimes_R R$ we have that

$$0 \to A \otimes_{R_S} R_S \to B \otimes_{R_S} R_S \to C \otimes_{R_S} R_S \to 0$$

is also an exact sequence. By restriction of scalars we may consider this to be a sequence of R-modules. M is assumed to be flat as an R-module, therefore the following will also be exact:

$$0 \to (A \otimes_{R_S} R_S) \otimes_R M \to (B \otimes_{R_S} R_S) \otimes_R M \to (C \otimes_{R_S} R_S) \otimes_R M \to 0.$$

Finally, the above tensors are "associative", therefore

$$0 \to A \otimes_{R_S} (R_S \otimes_R M) \to B \otimes_{R_S} (R_S \otimes_R M) \to C \otimes_{R_S} (R_S \otimes_R M) \to 0$$

is exact. The expression in parenthesis above is isomorphic to M_S thus we have

$$0 \to A \otimes_{R_S} M_S \to B \otimes_{R_S} M_S \to C \otimes_{R_S} M_S \to 0$$

and therefore M_S is flat.

Finally we show that M_S is finitely presentable if M is finitely presentable. Assume M is finitely presentable then there is an exact sequence

$$R^a \to R^b \to M \to 0$$

of free R-modules of finite rank a and b. Equivalently, M is finitely generated by b elements and the kernel of the corresponding R-module homomorphism $R^b \to M$ can be generated by a elements. Because the functor $(\_)_S$ preserves the sequences, we apply it to this exact sequence above, then we get:

$$(R^a)_S \to (R^b)_S \to M_S \to 0$$

And this is still an exact sequence, so we get that $M_S \cong (R^b)_S/(R^a)_S$. Because we have already shown $(R^a)_S$ and $(R^b)_S$ are free of finite rank if R^a and R^b are free of finite rank, we can conclude that M_S is finitely presentable.

$\square$