Exercise 7: *(Chriestenson, Strider) Suppose that*

$$0 \to \mathrm{Hom}_R(C, M) \xrightarrow{\psi^*} \mathrm{Hom}_R(B, M) \xrightarrow{\phi^*} \mathrm{Hom}_R(A, M)$$

is exact for every R-module M. Show that

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

is exact.

Proof: Let R be a commutative unital ring. Let A, B, C be R-modules, and $\phi \in \mathrm{Hom}_R(A, B)$, $\psi \in \mathrm{Hom}_R(B, C)$. Suppose that the Hom sequence listed above is exact for every R-module M.

If we let $M = \mathrm{coker}(\psi)$, then for the projection $\pi : C \to C/\mathrm{im}(\psi)$ we have that

$$\psi^*(\pi) = \pi\psi = 0.$$

Thus $\pi = 0$. This means that $\mathrm{im}(\psi) = C$.

If we let $M = C$, then we have that

$$\psi\phi = \phi^*(\psi) = \phi^*\psi^*(\mathrm{id}_C) = 0.$$

Thus

$$\mathrm{im}(\phi) \subseteq \ker(\psi).$$

If we let $M = \mathrm{coker}(\phi)$, then for the projection $\eta : B \to B/\mathrm{im}(\phi)$ we have that

$$\phi^*(\eta) = \eta\phi = 0.$$

Thus $\eta \in \ker(\phi^*) = \mathrm{im}(\psi^*)$. Let $\alpha \in \mathrm{Hom}_R(C, M)$ such that

$$\psi^*(\alpha) = \alpha\psi = \eta.$$

This implies that

$$\ker(\psi) \subseteq \ker(\eta) = \mathrm{im}(\phi).$$

We have shown that ψ is surjective, and $\mathrm{im}(\phi) = \ker(\psi)$. Therefore the sequence

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

is exact.